1. The space reflection (parity) operator changes \( \vec{x} \rightarrow \vec{x}' = -\vec{x} \) and \( t = x^0 \rightarrow x'^0 = x^0 \), which can be compactly written as:

\[
x'^\mu = a_\nu^\mu x^\nu \quad \text{where clearly} \quad a_\nu^\mu = g_\nu^\mu ,
\]

i.e., \( a_\nu^\mu \) is simply the Minkowski metric tensor. At the same time the Dirac spinor \( \psi \) will also transform:

\[
\psi(x) \rightarrow \psi'(x') = \psi'(ax) \equiv S \psi(x) .
\]

It can be shown in a few steps\(^1\) that for Dirac spinors the relation between a coordinate transformation \( a \) and the corresponding spinor transformation \( S \) is:

\[
a_\mu^\nu \gamma^\mu = S(a) \gamma^\nu S^{-1}(a) . \tag{1}
\]

(a) Show that for the parity transformation, i.e., for \( a_\nu^\mu = g_\nu^\mu \), the spinor transformation operator is given by \( S = P = \gamma^0 \), up to an arbitrary multiplicative phase factor without physical significance. [Hint: Reduce equation (1) to a set of (anti)commutation relations and find that \( S = \gamma^0 \) satisfies them.]

(b) Having established that the spinor parity operator \( P = \gamma^0 \), show that for particles (antiparticles) at rest, the Dirac spinors \( u^{(1,2)} \) (\( v^{(1,2)} \)) are parity eigenstates. Find the parity of \( u \) and \( v \), and show that they are opposite. This is the formal basis for the assertion made in class that fermions and antifermions have opposite intrinsic parity.

2. Prove the following basic trace theorems (with \( \phi = \gamma_\mu a_\mu \), and \( a \cdot b = a_\mu b_\mu \)):

\[
\begin{align*}
Tr\{\text{product of odd no. of } \gamma \text{ matrices}\} & \equiv 0 , \\
Tr\{\phi \phi\} & = 4a \cdot b , \\
Tr\{\phi \phi \phi \phi\} & = 4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)] .
\end{align*}
\]

3. Fill in the steps in deriving the angular distribution of elastic \( e^- + \mu^- \rightarrow e^- + \mu^- \) scattering at very high energies (i.e., with \( m_e, M_\mu \rightarrow 0 \)) in the center-of-momentum system:

\[
\frac{d\sigma}{d\Omega} = \frac{\hbar^2 c^2 a^2}{8 E^2} \frac{1}{\sin^4 \frac{\theta}{2}} \left( 1 + \cos \frac{\theta}{2} \right) , \quad \text{starting from} \quad |M|^2 = 2g_e^4 s^2 + u^2 .
\]

Use this result to plot the CM angular distribution at \( \sqrt{s} = 20 \text{ GeV} \). Compare this plot to the angular distribution of the same process at \( \sqrt{s} = 0.2 \text{ GeV} \) where you can no longer

\(^1\)See Bjorken and Drell, Relativistic Quantum Mechanics, chapter 2.
neglect the muon mass $M$. Use appropriate units, such as nb. Comment on the difference. You are encouraged to use computers in order to produce the plots.

4. Helicity, the projection of a particle’s spin on its momentum direction, $\mathbf{\hat{S}} \cdot \mathbf{\hat{p}} = \mathbf{\hat{S}} \cdot (\mathbf{\hat{p}}/|\mathbf{\hat{p}}|)$ is clearly defined only for a moving particle, and is a Lorentz invariant only for a massless particle. Righthanded and lefthanded Dirac fermions satisfy, respectively: $\mathbf{\hat{S}} \cdot \mathbf{\hat{p}} \psi_L = -\psi_L$ and $\mathbf{\hat{S}} \cdot \mathbf{\hat{p}} \psi_R = +\psi_R$.

(a) Show that $L = \frac{1}{2}(1 - \gamma^5)$ and $R = \frac{1}{2}(1 + \gamma^5)$ are projection operators, i.e., that $L^2 \equiv L$ and $R^2 \equiv R$.

(b) Show that $L$ and $R$, as defined above, represent the left- and right-handed helicity projection operators, respectively, for Dirac spinors, i.e., $\psi_L = L \psi$ and $\psi_R = R \psi$.

5. The familiar Dirac-Pauli matrices $\gamma^\mu$ are convenient because they diagonalize the particle’s energy. The Weyl, or chiral representation makes a different choice (which diagonalizes helicity):

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \text{ with } k = 1, 2, 3.$$  

(a) Show that the Weyl $\gamma$ matrices satisfy the same anticommutation relations as the Dirac-Pauli matrices: $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.

(b) Evaluate $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ in the Weyl representation and show that it is diagonal.

(c) Decomposing, as before, the Dirac spinor into two two-component spinors $\phi$ and $\chi$: $\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$, write down the Dirac equations for $\phi$ and $\chi$. Show that they decouple for a massless particle, $m = 0$.

(d) Show that the massless Weyl spinors $\phi$ and $\chi$ are left- and right-handed eigenstates, respectively, of the $L$ and $R$ projection operators from the preceding problem.