Notes on the Cumulative Distribution Function (cdf)
Math 310/510, Chapter 4, Fall 2003, C. Dunkl

These notes expand on the discussion in the text. The topic is very important and should be considered as exam material (at least for comprehension and usage in problems).

A random variable $X$ is a probability experiment whose outcomes are real numbers; the sample space is all of $\mathbb{R}$ or a subset like $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$ or an interval $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, and the events are intervals or unions of intervals. It is convenient to use $S = \mathbb{R}$ (with the possibility that some values are never taken on). All the probability information about $X$ is encoded in its cumulative distribution function (cdf) $F(x)$, defined by

$$F(x) = P\{X \leq x\} = P\{(-\infty, x]\};$$

to say $X \leq x$ is equivalent to the occurrence of the event $(-\infty, x] = \{r \in \mathbb{R} : r \leq x\}$. The axioms of probability have some important consequences for the properties of a cumulative distribution function. Recall that if $E, F$ are events and $E \subset F$ then $0 \leq P(E) \leq P(F) \leq 1$, and if $E_1, E_2, \ldots$ are a countable collection of pairwise disjoint (mutually exclusive, $i \neq j$ implies $E_i \cap E_j = \emptyset$) events then

$$\lim_{n \to \infty} \sum_{i=1}^{n} P(E_i) = \sum_{i=1}^{\infty} P(E_i) = P\left(\bigcup_{i=1}^{\infty} E_i\right).$$

The main properties of $F(x)$ are:

1. $0 \leq F(x) \leq 1$ for all $x$
2. $x_1 < x_2$ implies $F(x_1) \leq F(x_2)$ (this is called the “increasing” property); valid because $(-\infty, x_1] \subset (-\infty, x_2]$
3. $x_1 < x_2$ implies $P\{x_1 < X \leq x_2\} = P\{(x_1, x_2]\} = F(x_2) - F(x_1)$. Compare this to the fundamental theorem of calculus (for calculating definite integrals). Proof: note $(-\infty, x_2] = (-\infty, x_1] \cup (x_1, x_2]$, and this is a disjoint union, hence $P\{(-\infty, x_2]\} = P\{(-\infty, x_1]\} + P\{(x_1, x_2]\}$, thus $P\{(x_1, x_2]\} = P\{(-\infty, x_2]\} - P\{(-\infty, x_1]\} = F(x_2) - F(x_1)$.

4. $\lim_{x \to \infty} F(x) = 1$. Proof: let $\{x_n\}$ be a sequence increasing to $\infty$, $(x_0 = 0 < x_1 < x_2 < \ldots)$ for example $x_n = n$; the set $\{x : x > 0\} = \bigcup_{n=1}^{\infty} (x_{n-1}, x_n]$, by axiom

$$P\{X > 0\} = 1 - P\{X \leq 0\} = 1 - F(0)$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} P\{(x_{i-1}, x_i]\} = \lim_{n \to \infty} \sum_{i=1}^{n} (F(x_i) - F(x_{i-1}))$$

$$= \lim_{n \to \infty} (F(x_n) - F(0)) = \lim_{n \to \infty} F(x_n) - F(0);$$

this is called a telescoping sum.
5. \( \lim_{x \to -\infty} F(x) = 0 \). Proof: let \( \{x_n\} \) be a sequence decreasing to \(-\infty\), \((x_0 = 0 > x_1 > x_2 > \ldots)\) for example \( x_n = -n \); the set \( \{x : x \leq 0\} = \bigcup_{n=1}^{\infty} [x_n, x_{n-1}] \); then \( F(0) = P\{X \leq 0\} = \lim_{n \to \infty} \sum_{i=1}^{n} P\{x_i, x_{i-1}\} = \lim_{n \to \infty} (F(0) - F(x_n)) = F(0) - \lim_{n \to \infty} F(x_n) \).

6. \( F \) is “right-continuous” at every point; this means if \( \{x_n\} \) is a sequence converging to \( x \) from above \((x < \ldots < x_2 < x_1 < x_0)\) then \( \lim_{r \to x^+} F(r) = F(x) \) (equivalent notation: \( \lim_{r \to x^-} F(r) = F(x) \)). Proof: by

\[ (3) F(x_0) - F(x) = P\{(x, x_0]\} = P\{\bigcup_{i=1}^{\infty} (x_i, x_{i-1}]\} = \lim_{n \to \infty} (F(x_0) - F(x_n)) \].

7. \( F \) has a left-hand limit at every point and

\[ P\{X = x\} = F(x) - \lim_{r \to x^-} F(r) \].

Proof: Let \( x_0 < x_1 < x_2 < \ldots \) be a sequence converging to \( x \) from below (for example: \( x_n = x - 1/10^n \)); then the open interval \((x_0, x)\) is the countable disjoint union \( \bigcup_{i=1}^{\infty} (x_{i-1}, x_i] \) and \( P\{(x_0, x]\} = \lim_{n \to \infty} \sum_{i=1}^{n} P\{(x_{i-1}, x_i]\} = \lim_{n \to \infty} \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = \lim_{r \to x^-} F(r) - F(x_0) \).

The interval \((x_0, x]\) is the disjoint union of \((x_0, x)\) and \( \{x\} \) thus \( P\{X = x\} = P\{(x_0, x]\} - P\{(x_0, x)\} = F(x) - \lim_{r \to x^-} F(r) \).

Part (7) shows that the probability of an open interval uses the left-hand limit of \( F \) at the upper end-point, that is, \( P\{a < X < b\} = \lim_{r \to b^-} F(r) - F(a) \) (compare to formula (3)). Thus \( F \) is continuous at a point \( x \) (left and right limits are the same, or there is no jump discontinuity) if and only if \( P\{X = x\} = 0 \). If \( F \) is continuous on some interval \((a, b)\) and is differentiable there (the derivative \( f(x) = \frac{d}{dx} F(x) \) is called the “probability density function” (pdf)) then for \( a < x_1 < x_2 < b \) we have \( P\{x_1 \leq X \leq x_2\} = P\{x_1 < X < x_2\} = \int_{x_1}^{x_2} f(x) \, dx \). A random variable whose cdf has jump discontinuities and zero derivative between the jumps (constant) is called a discrete random variable. A random variable whose cdf has no jump discontinuities and is everywhere differentiable is called a continuous random variable. It is possible to have mixtures of the two properties.