Math 312/512 Assignment #1, Ch. 7, Jan. 22, 2004
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1) Find MLE for \( \theta \), given random sample from exponential density

\[
f(x) = \begin{cases} 
  e^{-(x-\theta)} & \text{for } x \geq \theta, \\
  0 & \text{for } x < \theta 
\end{cases}
\]

observe \( x_1, x_2, \ldots, x_n \), likelihood function is \( f(x_1, x_2, \ldots, x_n | \theta) = \prod_{i=1}^{n} e^{-(x_i - \theta)} \) with each \( x_i \geq \theta \); then the log-likelihood function is \( L(x_1, x_2, \ldots, x_n | \theta) = \log f = -n \sum_{i=1}^{n} (x_i - \theta) = n\theta - \sum_{i=1}^{n} x_i \). This is an increasing function in \( \theta \) (derivative = \( n \)) so has its maximum at the upper endpoint of its domain of definition: \( \theta \leq x_i \) for each \( i \), equivalent to \( \theta \leq \min_i \{ x_i \} \); thus the MLE for \( \theta \) is \( \min_i \{ x_i \} \).

2) Find MLE for \( \theta \), given random sample from two-sided exponential density

\[
f(x) = \frac{1}{2} e^{-|x-\theta|},
\]
similarly to (1) the likelihood function is \( f(x_1, x_2, \ldots, x_n | \theta) = 2^{-n} \prod_{i=1}^{n} e^{-|x_i - \theta|} \) and \( L(x_1, x_2, \ldots, x_n | \theta) = \log f = -n \log 2 - \sum_{i=1}^{n} |x_i - \theta| \). Relabel the data points so that \( x_1 < x_2 < x_3 < \ldots < x_n \). When \( x_m \leq \theta \leq x_{m+1} \) we have

\[
|x_{j} - \theta| = \begin{cases} 
  x_{j} - \theta & \text{for } j \geq m + 1 \\
  \theta - x_{j} & \text{for } j \leq m 
\end{cases}
\]

and \( -\sum_{i=1}^{m} |x_i - \theta| = -\sum_{i=1}^{m} (\theta - x_i) - \sum_{i=m+1}^{n} (x_j - \theta) = (-m + n - m) \theta + \sum_{i=1}^{m} x_i - \sum_{i=m+1}^{n} x_i \). Note the slope is \( n - 2m \). When \( 2m < n \) the function \( L(\theta) \) is increasing, when \( 2m > n \) the function \( L(\theta) \) is decreasing: if \( n \) is even \( L \) has slope 0 between \( x_{n/2} \) and \( x_{n/2+1} \) (so the MLE is the median, \( (x_{n/2} + x_{n/2+1})/2 \)); if \( n \) is odd the slopes have values \( 3, 1, -1, -3, \ldots \) and the maximum value is at the corner \( x_{(n+1)/2} \). The MLE for \( \theta \) is \( x_{(n+1)/2} \), again the median of \( x_1, x_2, \ldots, x_n \).

3) Random sample \( x_1, x_2, \ldots, x_n \) from \( N(\mu, \sigma^2) \) with \( \mu \) known. Use the calculations on page 221, replace \( \frac{\partial}{\partial \mu} \) by \( \frac{d}{d\mu} \), obtain the MLE for \( \sigma^2 \) is \( \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2 \). Further \( E\left[ \hat{\sigma}^2 \right] = \frac{1}{n} \sum_{i=1}^{n} E\left[ (x_i - \mu)^2 \right] = \frac{1}{n} \sum_{i=1}^{n} \sigma^2 = \frac{1}{n} (n\sigma^2) = \sigma^2 \).

4) There are two independent measurements, for \( X \) and \( \theta \); a model for measurements with random errors is usually the following: the measurement is a random variable \( X = \mu + \varepsilon \) where \( \mu \) is the actual value to be measured and \( \varepsilon \) denotes the measurement error, typically with the distribution \( N(0, \sigma^2) \). Given several independent observations the MLE for \( \mu \) is exactly the sample mean (more precisely, the observations are \( \mu + \varepsilon_1, \mu + \varepsilon_2, \ldots, \mu + \varepsilon_n \), then \( \frac{1}{n} \sum_{i=1}^{n} (\mu + \varepsilon_i) = \mu + \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i = \mu + \overline{\varepsilon} \); and \( E[\overline{\varepsilon}] = 0, \text{ var (} \overline{\varepsilon} \text{) =} \frac{\sigma^2}{n} \). In this problem, find the average \( \overline{X}, \theta \) values, estimate the height as \( \overline{X} \tan \theta \).