A New Approach to Orthogonal Polynomials in the Conditional Hazard Function

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Our specification for the conditional hazard function is

\[
\log h(t \mid X_1, X_2) = \lambda_1(z_1) + \lambda_2(z_2, t)
\]

where

\[
\lambda_1(z_1) = \sum_{i=0}^{M_1} a_i p_i(z_1),
\]

\[
\lambda_2(z_2, t) = \sum_{i=0}^{M_2} \sum_{j=0}^{M_2-i} b_{ij} p_i(z_2) p_j(t), \text{ and}
\]

\[
z_k = X_k \beta_k \text{ for } k = 1, 2
\]

with orthogonal polynomials \(p_i(\bullet)\). For sake of notation, we say that a set of polynomials is orthogonal with respect to \(f(\bullet)\) if

\[
\int p_i(z) p_j(z) f(z) \, dz = \delta_{ij}
\]

where

\[
\delta_{ij} = 1 (i = j).
\]

Our first method of constructing orthogonal polynomials was to pick an arbitrary density \(f(\bullet)\), which implied orthogonal polynomials. This did not work because, conditional on \(\beta = (\beta_1, \beta_2)\), the density of \(z = (z_1, z_2)\) was changing and was never \(f(\bullet)\). Our present method of constructing orthogonal polynomials was to adjust \(z_k\) by a normalization factor so that, at least, it had the right variance implied by \(f(\bullet)\). This was very ugly, and it did not work anyway, probably because it added too much nonlinearity to the problem.

We were thinking about this the wrong way. Instead of trying to adjust \(z\) to \(f(\bullet)\), we should be trying to adjust \(f(\bullet)\) to \(z\). This is described in detail below:
Conditional on a guess of $\beta$, we can evaluate $z$ for each observation, and, therefore, we can compute an estimate of the $k$th moment of $z_j$:

$$\hat{m}_{jk} = \frac{1}{n} \sum_{i=1}^{n} z_{ij}^k.$$ 

Note that equation (1) is equivalent to

$$E[p_i(z)p_j(z)] = \delta_{ij}.$$ 

If we write

$$p_i(z) = \sum_{k=0}^{i} \gamma_{ik} z^k,$$

then we can solve for $\gamma_i = (\gamma_{i0}, \gamma_{i1}, \ldots, \gamma_{ii})$ first by solving

$$C\phi = c$$

for $\phi$ where

$$\phi' = (\phi_{i0}, \phi_{i1}, \ldots, \phi_{ii-1}),$$

$$c = \begin{pmatrix}
-\sum_{k=0}^{i} \gamma_{0k} \hat{m}_i^{k+1} \\
-\sum_{k=0}^{i} \gamma_{1k} \hat{m}_i^{k+1} \\
\vdots \\
-\sum_{k=0}^{i-1} \gamma_{i-1k} \hat{m}_i^{k+1}
\end{pmatrix},$$

and

$$C = \begin{pmatrix}
\sum_{k=0}^{0} \gamma_{0k} \hat{m}_k & \sum_{k=0}^{0} \gamma_{0k} \hat{m}_1^{k+1} & \cdots & \sum_{k=0}^{0} \gamma_{0k} \hat{m}_{i-1}^{k+1} \\
\sum_{k=0}^{1} \gamma_{1k} \hat{m}_k & \sum_{k=0}^{1} \gamma_{1k} \hat{m}_1^{k+1} & \cdots & \sum_{k=0}^{1} \gamma_{1k} \hat{m}_{i-1}^{k+1} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{k=0}^{i-1} \gamma_{i-1k} \hat{m}_k & \sum_{k=0}^{i-1} \gamma_{i-1k} \hat{m}_1^{k+1} & \cdots & \sum_{k=0}^{i-1} \gamma_{i-1k} \hat{m}_{i-1}^{k+1}
\end{pmatrix},$$

then by setting

$$s^2 = 1 + \sum_{k=0}^{i-1} \phi_{ik}^2,$$

and then setting

$$\gamma_{ik} = \begin{cases}
1/s & \text{if } k = i \\
\phi_{ik}/s & \text{if } k < i
\end{cases}.$$ 

Further note that we can use a set of orthogonal polynomials (with good linear independence properties) to make a good guess of the next set of parameters, but then, given a new $\beta$, we can construct new orthogonal polynomials without changing the value of the likelihood contribution for any observation. Let $\beta^{(r)}$
be the \( r \)th guess of \( \beta \), and let \( p^{(r)}(\bullet) \) be the corresponding set of orthogonal polynomials. Then, for each observation \( i \), we can write

\[
\lambda_1(z_1) = \sum_{i=0}^{M_1} a_i^{(r)} p_i^{(r)}(z_1) \tag{2}
\]

\[
= \sum_{i=0}^{M_1} \sum_{k=0}^{i} a_i^{(r)} \gamma_{ik}^r z_1^k
\]

\[
= \sum_{k=0}^{i} \left( \sum_{i=k}^{M_1} a_i^{(r)} \gamma_{ik}^r \right) z_1^k
\]

\[
= \sum_{k=0}^{M_1} a_k^{(r)} z_1^k
\]

with

\[
a_k^{(r)} = \sum_{i=k}^{M_1} a_i^{(r)} \gamma_{ik}^r.
\]

Now, given a new guess \( \beta^{(r+1)} \) and corresponding \( p^{(r+1)}(\bullet) \), we can translate equation (2) into

\[
\lambda_1(z_1) = \sum_{i=0}^{M_1} a_i^{(r+1)} p_i^{(r+1)}(z_1)
\]

\[
= \sum_{i=0}^{M_1} \sum_{k=0}^{i} a_i^{(r+1)} \gamma_{ik}^{(r+1)} z_1^k
\]

where we can iteratively calculate \( a_i^{(r+1)} \):

\[
\frac{a_i^{(r+1)}}{a_i^{(r)}} \frac{\gamma_{M_1}^{M_1+1}}{\gamma_{M_1+1}^{M_1+1}} = \frac{a_i^{(r+1)}}{a_i^{(r)}} \frac{\gamma_{M_1}^{M_1+1}}{\gamma_{M_1+1}^{M_1+1}}
\]

\[
\Rightarrow \frac{a_i^{(r+1)}}{a_i^{(r)}} = \frac{a_i^{(r+1)}}{a_i^{(r)}} \frac{\gamma_{M_1}^{M_1+1}}{\gamma_{M_1+1}^{M_1+1}}
\]

and

\[
\sum_{k=i}^{M_1} \frac{a_i^{(r+1)}}{a_i^{(r)}} \frac{\gamma_{k}^{k}}{\gamma_{k+1}^{k+1}} z_1^k = \sum_{k=i}^{M_1} \frac{a_i^{(r+1)}}{a_i^{(r)}} \frac{\gamma_{k}^{k}}{\gamma_{k+1}^{k+1}} z_1^k
\]

\[
\Rightarrow \frac{a_i^{(r+1)}}{a_i^{(r)}} = \frac{\sum_{k=i}^{M_1} a_i^{(r+1)} \frac{\gamma_{k}^{k}}{\gamma_{k+1}^{k+1}} z_1^k - \sum_{k=i}^{M_1} a_i^{(r+1)} \frac{\gamma_{k}^{k}}{\gamma_{k+1}^{k+1}} z_1^k}{\gamma_{i+1}^{(r+1)}}
\]

for \( 0 \leq i < M_1 \). The set \( \left\{ \frac{a_i^{(r+1)}}{a_i^{(r)}} \right\}_{i=0}^{M_1} \) replaces \( \left\{ \frac{a_i^{(r+1)}}{a_i^{(r)}} \right\}_{i=0}^{M_1} \) with no change in the log likelihood contribution for any observation.
We can do an analogous transformation for

$$\lambda_2 (z_2, t) = \sum_{i=0}^{M_2} \sum_{j=0}^{M_2-i} b_{ij}^{(r)} p_i (z_2) p_j (t)$$

$$= \sum_{i=0}^{M_2} \sum_{j=0}^{M_2-i} b_{ij}^{(r)} \left[ \sum_{k=0}^{i} \gamma_{z ik}^{(r)} \sum_{k=0}^{j} \gamma_{izj}^{(r)} k^k \right].$$

Now, given a new guess $\beta^{(r+1)}$ and corresponding $p^{(r+1)} (\bullet)$, we can translate equation (2) into

$$\lambda_2 (z_2, t) = \sum_{i=0}^{M_2} \sum_{j=0}^{M_2-i} \tilde{b}_{ij}^{(r+1)} p_{zi}^{(r+1)} (z_2) p_{ij}^{(r+1)} (t)$$

$$= \sum_{i=0}^{M_2} \sum_{j=0}^{M_2-i} \tilde{b}_{ij}^{(r+1)} \left[ \sum_{k=0}^{i} \gamma_{z ik}^{(r+1)} z_2^k \sum_{k=0}^{j} \gamma_{izj}^{(r+1)} k^k \right].$$

where we can iteratively calculate $\tilde{b}_{ij}^{(r+1)}$:

$$\tilde{b}_{M20}^{(r+1)} \gamma_{z M2 M2}^{(r+1)} \gamma_{22}^{(r+1)} = b_{M20}^{(r)} \gamma_{z M2 M2}^{(r)} \gamma_{22}^{(r)}$$

$$\Rightarrow \tilde{b}_{M20}^{(r+1)} = \frac{b_{M20}^{(r)} \gamma_{z M2 M2}^{(r)} \gamma_{22}^{(r)}}{\gamma_{z M2 M2}^{(r+1)}}$$

$$\tilde{b}_{0 M2}^{(r+1)} \gamma_{z 00}^{(r+1)} \gamma_{0 M2 M2}^{(r+1)} = b_{0 M2}^{(r)} \gamma_{z 00}^{(r)} \gamma_{0 M2 M2}^{(r)}$$

$$\Rightarrow \tilde{b}_{0 M2}^{(r+1)} = \frac{b_{0 M2}^{(r)} \gamma_{z 00}^{(r)} \gamma_{0 M2 M2}^{(r)}}{\gamma_{z 00}^{(r+1)}}$$

and

$$\sum_{i=k_s}^{M_2} \sum_{j=k_t}^{M_2-i} \tilde{b}_{ij}^{(r+1)} \gamma_{z ik}^{(r+1)} \gamma_{ij}^{(r+1)} z_2^k k^k = \sum_{i=k_s}^{M_2} \sum_{j=k_t}^{M_2-i} b_{ij}^{(r)} \gamma_{z ik}^{(r)} \gamma_{ij}^{(r)} z_2^k k^k$$

$$\Rightarrow \tilde{b}_{k_s k_t}^{(r+1)} = \frac{\sum_{i=k_s}^{M_2} \sum_{j=k_t}^{M_2-i} b_{ij}^{(r)} \gamma_{z ik}^{(r)} \gamma_{ij}^{(r)} z_2^k k^k}{\gamma_{z k_s k_t}^{(r+1)} \gamma_{t k_t k_t}^{(r+1)}}.$$

Equation (3) can be used iteratively by solving for $\tilde{b}_{M2-1, 1}^{(r+1)}, \tilde{b}_{1, M2-1}^{(r+1)}, \tilde{b}_{M2-1, 0}^{(r+1)}, \tilde{b}_{0, M2-1}^{(r+1)}, \tilde{b}_{M2-2, 2}^{(r+1)}, \tilde{b}_{2, M2-2}^{(r+1)}, \tilde{b}_{M2-2, 1}^{(r+1)}, \tilde{b}_{1, M2-2}^{(r+1)}, \tilde{b}_{M2-2, 0}^{(r+1)}, \tilde{b}_{0, M2-2}^{(r+1)}, \ldots, \tilde{b}_{0, 0}^{(r+1)}.$