Smooth Simulation of Player 2 Acceptance
Probabilities

Steven Stern
December 30, 2005

1 Simulation of Probability of Player 2 Acceptance

Our problem assumes that

\[
\left( \begin{array}{c}
\log \beta_{21} \\
\log \beta_{22}
\end{array} \right) \sim iidN\left( \left( \begin{array}{c} a \\
-a
\end{array} \right), \left( \begin{array}{cc} 1 & \rho \\
\rho & 1
\end{array} \right) \right).
\]

The term that needs to be simulated is

\[
p_k = \Pr \left[ \sum_{j \in A_k} \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) \geq 0 \right].
\]

Define

\[
v_j = \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right)
\]

\Rightarrow p_k = \Pr \left[ \sum_{j \in A_k} v_j \beta_{2j} \geq 0 \right].

Let

\[
\overline{j} = \arg \max_{j \in A_k} v_j,
\]
\[
\underline{v} = v_{\overline{j}},
\]
\[
\overline{j} = \arg \min_{j \in A_k} v_j,
\]
\[
\underline{v} = v_{\overline{j}}.
\]
If
\[ \tau \leq 0, \, p_k = 0 \]
\[ \tau \geq 0, \, p_k = 1. \]

Write
\[
p_k = \Pr \left[ \sum_{j \in A_k, j \neq j_j} \nu_j \beta_{2j} + \nu \beta_{2j} + \nu \beta_{2j} \geq 0 \right]
\]
\[
= \Pr \left[ \alpha + \nu \beta_{2j} + \nu \beta_{2j} \geq 0 \right]
\]
where
\[ \alpha = \sum_{j \in J} \nu_j \beta_{2j} \]
and \( J = \{ j \in A_k, j \neq j_j \} \). Define \( \beta_J \) to be the vector of \( \{ \beta_j \}_{j \in J} \) with density \( f_J (\cdot) \). Note that it is straightforward to simulate \( \beta_J \) and therefore \( \alpha \). Now write
\[
p_k = \int \Pr \left( \alpha + \nu \beta_{2j} + \nu \beta_{2j} \geq 0 \mid \beta_J \right) f_J (\beta_J) \, d\beta_J
\]
\[
= \int \left[ \Pr \left( \alpha + \nu \beta_{2j} \geq 0 \mid \beta_J \right)
+ \Pr \left( \alpha + \nu \beta_{2j} + \nu \beta_{2j} \geq 0 \mid \beta_J, \alpha + \nu \beta_{2j} < 0 \right) \cdot \Pr \left( \alpha + \nu \beta_{2j} < 0 \mid \beta_J \right) \right] f_J (\beta_J) \, d\beta_J
\]
\[
\approx \int \left[ \Pr \left( \log \beta_{2j} \leq \log \frac{-\alpha}{\nu} \mid \beta_J \right)
\right. \\
+ \Pr \left( \log \beta_{2j} \geq \log \frac{-\alpha - \nu \beta_{2j}}{\nu} \mid \beta_J, \alpha + \nu \beta_{2j} < 0 \right) \cdot \Pr \left( \log \beta_{2j} > \log \frac{-\alpha}{\nu} \mid \beta_J \right) \right] f_J (\beta_J) \, d\beta_J
\]
\[
= \int \left[ \Phi \left( \log \frac{-\alpha}{\nu} + \frac{\mu_{j | \beta_J}}{\sqrt{\Omega_{j | \beta_J}}} \right)
+ \Phi \left( -\log \frac{-\alpha - \mu_{j | \beta_J}}{\nu} \right) \Phi \left( -\log \frac{-\alpha - \nu \beta_{2j}}{\nu} + \frac{\mu_{j | \beta_J, \alpha + \nu \beta_{2j}}}{\sqrt{\Omega_{j | \beta_J, \alpha + \nu \beta_{2j}}}} \right) \right] f_J (\beta_J) \, d\beta_J
\]
where \( \left( \mu_{j|\beta_j}, \sqrt{\Omega_{jj|\beta_j}} \right) \) are the mean and standard deviation of \( \log \beta_{j\tau} \) conditional on \( \beta_J \) and \( \left( \mu_{j|\beta_j,\alpha+\omega^2_{j\tau}}, \sqrt{\Omega_{jj|\beta_j,\alpha+\omega^2_{j\tau}}} \right) \) are the mean and standard deviation of \( \log \beta_{j\tau} \) conditional on \( \beta_J \) and \( \alpha + \omega^2_{j\tau} \). Note that the step in equation (2) is not true but in the same sense as GHK.\(^1\) This suggests a GHK-type algorithm:

1. Simulate \( \beta_J \), and construct \( \alpha \).
2. Compute \( \mu_{j|\beta_j} \).
3. Compute
   \[
   \Phi_1 = \Phi \left( \frac{\log \frac{-\alpha}{v} - \mu_{j|\beta_j}}{\sqrt{\Omega_{jj|\beta_j}}} \right)
   \]
   and
   \[
   \Phi_2 = \Phi \left( \frac{-\log \frac{-\alpha}{v} + \mu_{j|\beta_j}}{\sqrt{\Omega_{jj|\beta_j}}} \right) = 1 - \Phi \left( \frac{\log \frac{-\alpha}{v} - \mu_{j|\beta_j}}{\sqrt{\Omega_{jj|\beta_j}}} \right)
   \]
   using the simulated value of \( \beta_J \).
4. Simulate \( \log \beta_{2\tau} \mid \beta_J \), \( \log \beta_{2\tau} \geq \log \frac{-\alpha}{v} \).
5. Compute \( \alpha + v^2_{2\tau} \) using the simulated value of \( \log \beta_{2\tau} \).
6. Compute \( \mu_{j|\beta_j,\alpha+\omega^2_{2\tau}} \).
7. Compute
   \[
   \Phi_3 = \Phi \left( \frac{-\log \frac{-\alpha}{v} - \mu_{j|\beta_j,\alpha+\omega^2_{2\tau}}}{\sqrt{\Omega_{jj|\beta_j,\alpha+\omega^2_{2\tau}}} + \mu_{j|\beta_j,\alpha+\omega^2_{2\tau}}} \right)
   \]
   using the simulated values of \( \beta_J \) and \( \alpha + v^2_{2\tau} \).
8. Compute
   \[
   \Phi_1 + \Phi_2 \Phi_3.
   \]

We can show that this algorithm is an importance sampler and is thus an unbiased simulator of \( p_k \). Furthermore, by construction, the simulator is smooth in \( \{v_j\}_{j \in A_k} \) and is thus smooth in \( \{t_j\}_{j \in A_k} \). Finally we use antithetic acceleration (Geweke 1988) to reduce variance of our simulator.

\(^1\)Geweke (1991), Hajivassiliou (1990), and Keane (1994).
2 Simulation of Player 2’s Expected Value

We still have the same distribution for $\log \beta$ as in equation (1). But now we want to simulate

$$E_k = \int \left[ \sum_{j \in A_k} \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) \right] 1 \left[ \sum_{j \in A_k} \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) \geq 0 \right] f(\beta_2) d\beta_2$$

$$= \int \left( \sum_{j \in A_k} v_j \beta_{2j} \right) 1 \left( \sum_{j \in A_k} v_j \beta_{2j} \geq 0 \right) f(\beta_2) d\beta_2.$$ 

The frequency simulator,

$$\left( \sum_{j \in A_k} v_j \beta_{2j} \right) 1 \left( \sum_{j \in A_k} v_j \beta_{2j} \geq 0 \right)$$

where $\beta_2$ is drawn from the distribution in equation (1) is already continuous in $v$. However, the errors used to simulate equation (3) would not be the same as those used in the GHK simulator and thus the simulators would not be consistent with each other. The other problem with player 2’s simulated expected value is that the simulated $\beta_2$’s are no longer normalized such that $\sum_j \beta_{2j} = 4$.

The first problem can be dealt with by mimicking the GHK algorithm. In particular, define

$$R = \left\{ \beta_2 : \sum_{j \in A_k} \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) \geq 0 \right\},$$

and write the expected value for player 2 as

$$E_2 = \int_R \sum_{j \in A_k} \left[ \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) \right] dF(\beta_2) \quad (4)$$

$$= \int_R \left[ \sum_{j \in A_k} \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) + \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) \right]$$

$$+ \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) dF(\beta_2).$$
Now define

\[ R_1 = \left\{ \beta_2 : \sum_{j \in J} \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) < 0 \right\} \cap R; \]

\[ R_2 = \left\{ \beta_2 : \sum_{j \in J} \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) \geq 0, \right. \]

\[ \left. \sum_{j \in J} \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) + \beta_{2L} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) < 0 \right\} \cap R; \]

\[ R_3 = \left\{ \beta_2 : \sum_{j \in J} \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) \geq 0, \right. \]

\[ \left. \sum_{j \in J} \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) + \beta_{2L} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) \geq 0 \right\} \cap R. \]

Note that

\[ \bigcup_{m=1}^{3} R_m = R. \]

Now write equation (4) as

\[ = \sum_{m=1}^{3} \int_{R_m} \left[ \sum_{j \in J} \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) + \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) \right] \left( \beta_2 \right) \]

\[ + \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) \right] \left( \beta_2 \right) \]
The first two terms (for $m \leq 2$) can be written as

$$\int_{R_m} \left[ a \Phi \left( \frac{-\log \max (-a, 0) + \log \left( \frac{z \sigma + t_j}{\sigma \sigma} \right) + \mu}{\sigma} \right) \right] dF \left( \beta_J, \beta_{2J} \right)$$

(6)

where

$$\mu = E \left( \log \beta_{2J} \mid \log \beta_J, \log \beta_{2J} \right) ;$$

$$\sigma^2 = Var \left( \log \beta_{2J} \mid \log \beta_J, \log \beta_{2J} \right) ;$$

$$a = \sum_{\beta \in J} \beta_{\beta} \log \left( \frac{z_{2J} + t_j}{z_{2J}} \right) + \beta_{2J} \log \left( \frac{z_{2J} + t_j}{z_{2J}} \right).$$

---

2This is a generalization of the formulas in Heckman (1979). The integral we are interested in has the form,

$$\int_{x > \max (-a, 0)} \frac{(vx + a)}{\sigma} \phi \left( \frac{\log x - \mu}{\sigma} \right) dx.$$

Using the transformation of variables,

$$z = \log x \Rightarrow dz = \frac{dx}{x},$$

we can write our equation as

$$\int_{z > \max (-a, 0) - \log v} \frac{(ve^z + a)}{\sigma} \phi \left( \frac{z - \mu}{\sigma} \right) dz = a \Phi \left( -\log \max (-a, 0) + \log v + \mu \right)$$

$$+ v \int_{z > \max (-a, 0) - \log v} \exp \left\{ -\frac{(z^2 - 2z (\mu + \sigma^2) + \mu^2)}{2\sigma^2} \right\} dz$$

$$= a \Phi \left( -\log \max (-a, 0) + \log v + \mu \right)$$

$$+ v \exp \left\{ \mu + \frac{\sigma^2}{2} \right\} \Phi \left( \frac{-\log \max (-a, 0) + \log v + \mu + \sigma^2}{\sigma} \right).$$
Note that, when $m = 1$,

$$\sum_{j \in J} \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) \leq 0$$

$$\Rightarrow \sum_{j \in J} \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) + \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) \leq 0$$

because $\log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) \leq 0$ and $\beta_{2j} > 0$. However, when $m = 2$, we limit ourselves to

$$\beta_{2j} \geq \frac{\sum_{j \in J} \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right)}{\log \left( \frac{z_{2j} + t_j}{z_{2j}} \right)}.$$ 

The third term (for $m = 3$) can be written as

$$= \int_{R_3} \left[ \sum_{j \in J} \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) + \beta_{2j} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) \right. + \exp \left\{ \mu + \frac{\sigma^2}{2} \right\} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) \right] dF(\beta_j, \beta_{2j})$$

(7)

where

$$\mu = E \left[ \log \beta_{2j} \log \beta_j, \log \beta_{2j} \right];$$

$$\sigma^2 = Var \left[ \log \beta_{2j} \log \beta_j, \log \beta_{2j} \right].$$

Note that no conditioning is necessary in equation (7):

$$E \left[ \beta_{2j} \log \beta_j, \log \beta_{2j} \right] = \exp \left\{ \mu + \frac{\sigma^2}{2} \right\},$$

because, if the conditions for $\beta_j \in R_3$ are satisfied, then there are no restrictions on $\beta_{2j}$.

Solving the second problem, namely that we need to normalize $\beta_2$, means that we should be dividing the integrand in equation (5) by $\sum_{j} \beta_{2j} / 4$. However, this means that equations (6) and (7) are incorrect because there is now a $\beta_{2j}$ in the denominator. It is too difficult to analytically evaluate the new corrected integral. So, instead, we follow Berkovec and Stern (1991) and Keane (1994) and evaluate the expected value of the denominator separately. Essentially, we are approximating the expected value of a ratio by the ratio of expected values.
Berkovec and Stern (1991) and Keane (1994) suggest that the bias caused by this approximation is small in these kinds of problems. To do this, we need

\[
\frac{\int_{R_m} \left[ \sum_{j \in A_k} \beta_{2j} \right] dF(\beta_2)}{\int_{R_m} dF(\beta_2)}
\]

\[
= \frac{\int_{R_m} \left[ b + E \left( \beta_{2j} \mid \beta_J, \beta_{2j}, \beta_2 \in R_m \right) \right] dF(\beta_J, \beta_{2j})}{\int_{R_m} dF(\beta_2)}
\]

where

\[
b = \sum_{j \in J} \beta_{2j} + \beta_{2j}.
\]

For \( m \leq 2 \),

\[
b + E \left( \beta_{2j} \mid \beta_J, \beta_{2j}, \beta_2 \in R_m \right) = b \Phi \left( \frac{-\log \max (-a, 0) + \mu}{\sigma} \right) + \exp \left\{ \mu + \frac{\sigma^2}{2} \right\} \Phi \left( \frac{-\log \max (-a, 0) + \mu + \sigma^2}{\sigma} \right)
\]

where \((\mu, \sigma)\) are defined in equation (8) using the same argument as in equation (6). The denominator in equation (9) is

\[
\int \Phi \left( \frac{-\log \max (-a, 0) + \mu}{\sigma} \right) dF(\beta_J, \beta_{2j}).
\]

For \( m = 3 \), there are no conditions on \( \beta_{2j} \), so

\[
b + E \left( \beta_{2j} \mid \beta_J, \beta_{2j}, \beta_2 \in R_3 \right) = b + \exp \left\{ \mu + \frac{\sigma^2}{2} \right\},
\]

and the denominator is 1.

The analysis above suggests a GHK-like algorithm:

1. Simulate \( \beta_J \), and construct \( \alpha \).
2. Compute \( \mu_{j \mid \beta_J} \).
3. If \( \alpha \leq 0 \), simulate \( \beta_{2j} \).
4. If \( \alpha > 0 \), compute

\[
P_1 = \Pr \left[ \alpha + \beta_{2j} \log \left( \frac{z_{2j} + t_{ij}}{z_{2j}} \right) \leq 0 \right],
\]

\[
P_2 = 1 - P_1.
\]
and simulate
\[
\beta_{2 \perp}^{1} = \beta_{2 \perp} | \beta_j, \alpha + \beta_{2 \perp} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) \leq 0, \\
\beta_{2 \perp}^{2} = \beta_{2 \perp} | \beta_j, \alpha + \beta_{2 \perp} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) > 0.
\]

5. Using \(\alpha, \beta_{2 \perp}^{1}, \beta_{2 \perp}^{2}\), and equations (6), (7), (9), (10), (11), (12), and (13), compute
\[
4 \sum_{m=1,2} \frac{P_m}{E} \left[ \alpha + \beta_{2 \perp} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) + \beta_{2 \perp} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right) \mid \beta_j, \beta_{2 \perp}^{m} \right] \\
E \left[ \sum_{j \in J} \beta_{2 \perp} + \beta_{2 \perp}^{m} + \beta_{2 \perp} \mid \beta_j, \beta_{2 \perp}^{m} \right].
\]

3 Simulation of Probability of Player 2 Acceptance When There are Multiple Simultaneous Offers

When there are multiple simultaneous offers, let’s say indexed by \(m\), we can not use the GHK algorithm in Section 1 because there is no value of \(\beta_j\) and/or \(\beta_j\) conditional on \(\beta_j\) that can cause each offer to be accepted with a positive probability. However, all we need is that
\[
\sum_m P_{2m} V_{1m}
\]
is continuous in \(\{t_{mj}\}\) where
\[
V_{1m} = \sum_{k} 1 \left[ \sum_{j \in A_{mk}} \beta_{2j} \log \left( \frac{z_{2j} + t_{mj}}{z_{2j}} \right) \geq 0 \right] \sum_{j \in A_{mk}} \beta_{1j} \log \left( \frac{z_{1j} - t_{mj}}{z_{1j}} \right);
\]
\[
V_{2m} = \sum_{k} 1 \left[ \sum_{j \in A_{mk}} \beta_{2j} \log \left( \frac{z_{2j} + t_{mj}}{z_{2j}} \right) \geq 0 \right] \sum_{j \in A_{mk}} \beta_{2j} \log \left( \frac{z_{s} + t_{mj}}{z_{s}} \right);
\]
\[
P_{2m} = \Pr \left[ V_{2m} > V_{2n} \forall n \neq m \right].
\]
Thus it is sufficient for us to smooth only the conditional probabilities in a GHK algorithm corresponding to the offers with the likely greatest probabilities. In
particular, define

\[ v_{mj} = \log \left( \frac{z_{2j} + t_{mj}}{z_{2j}} \right); \]

\[ v_j = \sum_m v_{mj} \]

\[ \vec{j} = \arg \max_j v_j; \]

\[ \vec{\nu} = v_{\vec{j}}. \]

\[ J = \{ j : j \not= \vec{j} \}, \text{ and } \beta_J = \{ \beta_j : j \in J \}. \] We consider simulating \( \beta_J \) and then analytically computing each \( P_{2m} \) conditional on the simulated value of \( \beta_J \). The analytical step requires

\[ \Pr[V_{2m} > V_{2n} \quad \forall n \not= m \mid \beta_J]. \]

We can write equation (14) as

\[
V_{2m} = \sum_{k \not= k_m} 1 \left[ \sum_{j \in A_{mk}} \beta_{2j} v_{mj} \geq 0 \right] \sum_{j \in A_{mk}} \beta_{2j} v_{mj} + 1 \left[ \sum_{j \in A_{\vec{m}k_m}} \beta_{2j} v_{mj} \geq 0 \right] \sum_{j \in A_{\vec{m}k_m}} \beta_{xj} v_{mj} \tag{15}
\]

where \( k_m : \vec{j} \in A_{mk_m} \), and then we can further isolate \( \beta_{\vec{j}} \) by writing equation (15) as

\[
V_{2m} \left( \beta_{\vec{j}} \right) = \sum_{k \not= k_m} 1 \left[ \sum_{j \in A_{mk}} \beta_{2j} v_{mj} \geq 0 \right] \sum_{j \in A_{mk}} \beta_{2j} v_{mj} + 1 \left[ \sum_{j \in A_{\vec{m}k_m} \cap J} \beta_{2j} v_{mj} + \beta_{2j} v_{m\vec{j}} \geq 0 \right] \cdot \left[ \sum_{j \in A_{\vec{m}k_m} \cap J} \beta_{2j} v_{mj} + \beta_{2j} v_{m\vec{j}} \right]. \tag{16}
\]

Since we are interested in values of \( \beta_{\vec{j}} \) such that \( \max_m V_{2m} \left( \beta_{\vec{j}} \right) \geq 0 \), we should define \( \beta_{\vec{j}}^* \) as the value of \( \beta_{\vec{j}} \) such that

\[ \max_m V_{2m} \left( \beta_{\vec{j}}^* \right) = 0, \]
compute

$$\Pr \left( \beta_{2T} \geq \beta^*_{2T} \mid \beta_J \right).$$

Also, we need to find all critical values where \( \exists m_1, m_2 : \)

$$m_1, m_2 = \arg \max_m V_{2m} \left( \beta^*_{2T} \right); \quad (17)$$

where the best offer changes. We do this numerically using a combination line grid search and bisection routine. Consider the following GHK-like algorithm:

1. Simulate \( \beta_J \), and construct

$$a_{mk} = \sum_{j \in A_{mk} \cap J} \log \left( \frac{z_{2j} + t_j}{z_{2j}} \right)$$

for each \((k, m)\) combination.

2. Compute \( \mu_{\beta_J} \) (available in Section 5.1).

3. Compute \( \beta^*_{2T} \) and critical values defined by equation (17).

4. Compute probabilities based on critical values.

## 4 Simulation of Player 2’s Expected Value When There are Multiple Simultaneous Offers

We also need to compute Player 2’s expected value when there are multiple simultaneous offers. Using equation (16), we need only compute for two critical values, \( \left( \beta^*_{12T}, \beta^*_{22T} \right) \),

$$E \left( \beta_{2T} \mid \beta_J, \beta^*_{12T} \leq \beta_{2T} \leq \beta^*_{22T} \right) = \exp \left\{ \mu + \frac{\sigma^2}{2} \right\} \frac{\Phi \left( \frac{\beta^*_{22T} - (\mu + \sigma^2)}{\sigma} \right) - \Phi \left( \frac{\beta^*_{12T} - (\mu + \sigma^2)}{\sigma} \right)}{\Phi \left( \frac{\beta^*_{22T} - \mu}{\sigma} \right) - \Phi \left( \frac{\beta^*_{12T} - \mu}{\sigma} \right)}$$

where

$$\mu = E \left( \log \beta_{2T} \mid \log \beta_J \right);$$

$$\sigma^2 = \text{Var} \left( \log \beta_{2T} \mid \log \beta_J \right)$$

using the same analysis as above. Of course, we still need to normalize and can do so in the same way as in Section 2.
5 Appendices

5.1 Computation of Conditional Means and Variances

The joint density of $\log \beta_2$ is

$$
\begin{pmatrix}
\log \beta_{21} \\
\log \beta_{22} \\
\log \beta_{23} \\
\log \beta_{24}
\end{pmatrix}
\sim N
\begin{pmatrix}
1 + a \\
1 - a \\
1 + a \\
1 - a
\end{pmatrix},
\begin{pmatrix}
1 & \rho & 0 & 0 \\
\rho & 1 & 0 & 0 \\
0 & 0 & 1 & \rho \\
0 & 0 & \rho & 1
\end{pmatrix},
$$

and we need the conditional density of one element conditional on two others and one element conditional on three others. There are 12 cases to consider:

1. $j = 1, \bar{j} = 2$:

$$
\begin{align*}
\log \beta_{21} & \mid (\log \beta_{23}, \log \beta_{24}) \sim N [1 + a, 1] ; \\
\log \beta_{22} & \mid (\log \beta_{21}, \log \beta_{23}, \log \beta_{24}) \\
& \sim N [1 - a + \rho (\log \beta_{21} - 1 - a), 1 - \rho^2] .
\end{align*}
$$

2. $j = 1, \bar{j} = 3$:

$$
\begin{align*}
\log \beta_{21} & \mid (\log \beta_{22}, \log \beta_{24}) \sim N [1 + a + \rho (\log \beta_{22} - 1 + a), 1 - \rho^2] ; \\
\log \beta_{23} & \mid (\log \beta_{21}, \log \beta_{22}, \log \beta_{24}) \\
& \sim N [1 + a + \rho (\log \beta_{24} - 1 + a), 1 - \rho^2] .
\end{align*}
$$

3. $j = 1, \bar{j} = 4$:

$$
\begin{align*}
\log \beta_{21} & \mid (\log \beta_{22}, \log \beta_{23}) \sim N [1 + a + \rho (\log \beta_{22} - 1 + a), 1 - \rho^2] ; \\
\log \beta_{24} & \mid (\log \beta_{21}, \log \beta_{22}, \log \beta_{23}) \\
& \sim N [1 - a + \rho (\log \beta_{23} - 1 - a), 1 - \rho^2] .
\end{align*}
$$

4. $j = 2, \bar{j} = 1$:

$$
\begin{align*}
\log \beta_{22} & \mid (\log \beta_{23}, \log \beta_{24}) \sim N [1 - a, 1] ; \\
\log \beta_{21} & \mid (\log \beta_{22}, \log \beta_{23}, \log \beta_{24}) \\
& \sim N [1 + a + \rho (\log \beta_{22} - 1 + a), 1 - \rho^2] .
\end{align*}
$$

5. $j = 2, \bar{j} = 3$:

$$
\begin{align*}
\log \beta_{22} & \mid (\log \beta_{23}, \log \beta_{24}) \sim N [1 - a + \rho (\log \beta_{23} - 1 - a), 1 - \rho^2] ; \\
\log \beta_{23} & \mid (\log \beta_{22}, \log \beta_{22}, \log \beta_{24}) \\
& \sim N [1 + a + \rho (\log \beta_{24} - 1 + a), 1 - \rho^2]
\end{align*}
$$
6. $j = 2, \bar{j} = 4$:
\[
\begin{align*}
\log \beta_{22} & \sim N \left[ 1 - a + \rho (\log \beta_{21} - 1 - a), 1 - \rho^2 \right]; \\
\log \beta_{24} & \sim N \left[ 1 - a + \rho (\log \beta_{23} - 1 - a), 1 - \rho^2 \right].
\end{align*}
\]

7. $j = 3, \bar{j} = 1$:
\[
\begin{align*}
\log \beta_{23} & \sim N \left[ 1 + a + \rho (\log \beta_{24} - 1 + a), 1 - \rho^2 \right]; \\
\log \beta_{21} & \sim N \left[ 1 + a + \rho (\log \beta_{22} - 1 + a), 1 - \rho^2 \right].
\end{align*}
\]

8. $j = 3, \bar{j} = 2$:
\[
\begin{align*}
\log \beta_{23} & \sim N \left[ 1 + a + \rho (\log \beta_{24} - 1 + a), 1 - \rho^2 \right]; \\
\log \beta_{22} & \sim N \left[ 1 + a + \rho (\log \beta_{21} - 1 - a), 1 - \rho^2 \right].
\end{align*}
\]

9. $j = 3, \bar{j} = 4$:
\[
\begin{align*}
\log \beta_{23} & \sim N \left[ 1 + a, 1 \right]; \\
\log \beta_{24} & \sim N \left[ 1 - a + \rho (\log \beta_{23} - 1 - a), 1 - \rho^2 \right].
\end{align*}
\]

10. $j = 4, \bar{j} = 1$:
\[
\begin{align*}
\log \beta_{24} & \sim N \left[ 1 - a + \rho (\log \beta_{23} - 1 - a), 1 - \rho^2 \right]; \\
\log \beta_{21} & \sim N \left[ 1 + a + \rho (\log \beta_{22} - 1 + a), 1 - \rho^2 \right].
\end{align*}
\]

11. $j = 4, \bar{j} = 2$:
\[
\begin{align*}
\log \beta_{24} & \sim N \left[ 1 - a + \rho (\log \beta_{23} - 1 - a), 1 - \rho^2 \right]; \\
\log \beta_{22} & \sim N \left[ 1 + a + \rho (\log \beta_{21} - 1 - a), 1 - \rho^2 \right].
\end{align*}
\]

12. $j = 4, \bar{j} = 3$:
\[
\begin{align*}
\log \beta_{24} & \sim N \left[ 1 - a, 1 \right]; \\
\log \beta_{23} & \sim N \left[ 1 + a + \rho (\log \beta_{21} - 1 + a), 1 - \rho^2 \right].
\end{align*}
\]
5.2 Simulation of $\beta_j | \beta_J$, $\log \beta_j \geq \log \frac{-\alpha}{v}$

The density of $\beta_j | \beta_J$ can be written as

$$\beta_j | \beta_J \sim N(\mu, \sigma^2)$$

where $(\mu, \sigma^2)$ are given in Section 5.1. Then

$$u = \frac{\Phi\left(\frac{\beta_j - \mu}{\sigma}\right) - \Phi\left(\frac{\log \frac{-\alpha}{v} - \mu}{\sigma}\right)}{1 - \Phi\left(\frac{\log \frac{-\alpha}{v} - \mu}{\sigma}\right)}$$

is the distribution of $\beta_j | \beta_J$, $\log \beta_j \geq \log \frac{-\alpha}{v}$. Thus

$$\beta_j = \mu + \sigma \Phi^{-1}\left\{ u \left[ 1 - \Phi\left(\frac{\log \frac{-\alpha}{v} - \mu}{\sigma}\right)\right] + \Phi\left(\frac{\log \frac{-\alpha}{v} - \mu}{\sigma}\right) \right\}$$

has the desired distribution where $u \sim U(0,1)$.

References


