1 Lagrange Multiplier Test for State Effects

Our model (in equation (1)) is

\[ y_{nijt} = X_{nit} \beta_j + Z_{nijt} \gamma + \alpha_j y_{nijt-1} + \omega_{nijt} + \varepsilon_{nijt}. \]

In section 5.2, we consider a more general model,

\[ y_{nijt} = X_{nit} \beta_j + Z_{nijt} \gamma + \alpha_j y_{nijt-1} + \sum_s d_{nis} \tau_{is} + \omega_{nijt} + \varepsilon_{nijt}, \]

and we want to test

\[ H_0 : \tau_{is} = 0 \quad \forall s \]

against the general alternative. The complication is that there are a lot of singularities associated with state dummies \( d_{nis} \) and any variables in \( X_{nit} \) that are constant over time; i.e., all of the policy variables.

Instead of trying to carefully analytically determine all of the restrictions, we can achieve the same result more generally. Consider a problem,

\[ u_{K \times 1} \sim N(0, \Omega) \]
\[ A u = 0. \]

We can think of \( u \) as the vector of \( K \) score statistics from a Lagrange Multiplier test and \( A \) as the matrix of \( R \) restrictions imposed on \( u \) associated with state dummies and time-constant elements of \( X_{nit} \). We can decompose \( \Omega \) as

\[ \Omega = C \lambda C' \]

where \( C \) is the matrix of eigenvectors of \( \Omega \) and \( \lambda \) is a diagonal matrix with the nonnegative eigenvalues on the diagonal. The fact that there are \( R \) restrictions imposed on \( u \) implies that \( R \) of the eigenvalues in \( \lambda \) are zero, so we can write

\[ \lambda = \begin{pmatrix} 0 & 0' \\ R \times R & 0 \\ (K-R) \times R & D \\ 0 & (K-R) \times (K-R) \end{pmatrix} \]

and

\[ C = \begin{pmatrix} C_{11} & C_{12} \\ R \times R & R \times (K-R) \\ C_{21} & C_{22} \\ (K-R) \times R & (K-R) \times (K-R) \end{pmatrix} = \begin{pmatrix} C_1 \\ R \times K \\ C_2 \\ (K-R) \times K \end{pmatrix} \]

(where \( C_2 \) is the matrix of eigenvectors associated with the positive eigenvalues). We can ignore \( C_1 u \) because the restrictions imply that it is zero.

Thus, consider

\[ C_2 u \sim N(0, C_2 \Omega C_2'). \]

Note that

\[ C_2 \Omega C_2' = C_2 \lambda C' C_2 \]

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\[
\begin{align*}
&= C_2 \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \begin{pmatrix} 0 & 0' \\ 0 & D \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}' C_2' \\
&= \begin{pmatrix} C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} 0 & 0' \\ 0 & D \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}' \begin{pmatrix} C_{21}' \\ C_{22}' \end{pmatrix} \\
&= \begin{pmatrix} C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} C_{12}' C_{12}' C_{21}' + C_{22}' C_{22}' C_{21}' C_{22}' + C_{21}' C_{22}' C_{22}' C_{22}' + C_{22}' C_{22}' C_{22}' C_{22}' \\ C_{22}' C_{22}' C_{22}' C_{22}' \end{pmatrix} \\
&= C_{21}' C_{12}' C_{21} + C_{22}' C_{22}' C_{21} + C_{21}' C_{12}' C_{22}' C_{22} + C_{22}' C_{22}' C_{22}' C_{22} \\
&= \begin{pmatrix} C_{21} + C_{22} \end{pmatrix} D \begin{pmatrix} C_{12}' C_{21} + C_{22}' C_{22} \end{pmatrix} = D.
\end{align*}
\]

Then, under $H_0$,
\[
D^{-1/2} C_2 u \sim N (0, I_{K-R}),
\]
and
\[
\left( D^{-1/2} C_2 u \right)' \left( D^{-1/2} C_2 u \right) \sim \chi^2_{K-R}.
\]

Note that, instead of having to analytically determine all of the singularities in $u$, we need only count the number of positive eigenvalues.\(^1\)

For our problem, the sample test statistic is 1850.7, and it is distributed $\chi^2_{221}$ under $H_0$. This is statistically significant at any relevant size.

\(^1\)There is a roundoff error problem in that some zero eigenvalues will appear to be very small numbers. We use a rule of thumb that any eigenvalue less than 0.0001 is really zero.