An Empirical Dynamic Model of the Used Car Market

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1 Model

Let \( u_{ij} \) be the utility gross of cost that individual \( i \) gets from owning automobile \( j \) at time \( t \) where time is measured relative to the purchase time; \( t = 0 \) at the purchase time. Assume there is a finite number of brands \( J \). Assume that

\[
u_{ij} \sim iidF \left[ \theta(b_j); \sigma_n^2 \right]
\]

with support \([0,U]\). Let \( a_{jt} \) be the age of \( j \) at \( t \) and \( b_j \) be the brand of \( j \). Assume there are log operating costs of

\[
\log c_{jt} = \alpha_j + \lambda(b_j, a_{jt}), \tag{1}
\]

\[
\alpha_j \sim iidN \left[ \mu(b_j), \sigma^2 \right],
\]

and \( \lambda(b_j, a_{jt}) \) is a known function accounting for the effect of aging on operating costs with

\[
\frac{\partial \lambda(b_j,a)}{\partial a} > \lambda^*
\]

for some positive constant \( \lambda^* \). Prior to the purchase of \( j \), \( i \) does not know the value of \( \alpha_j \) but learns it immediately after purchase. Then the flow \( i \) receives from owning \( j \) at \( t \) is

\[
u_{ij} - c_{jt}.
\]

Let

\[
p_{jt} = p(b_j, a_{jt})
\]

be the price of automobile \( j \) at time \( t \). Then, the value to \( i \) of owning \( j \) at \( t \) is

\[
V(u_{ij}, \alpha_j, b_j, a_{jt}) = u_{ij} - c_{jt} + \beta \max \left[ W, p(b_j, a_{jt+1}) + W, V(u_{ij}, \alpha_j, b_j, a_{jt+1}) \right]
\]

where

\[
W = \beta \max_k \left\{ 0, E_{\alpha_k} V(u_{ik}, \alpha_k, b_k, a_{kt+1}) - p(b_k, a_{kt+1}) \mid V(v_{ik}, \alpha_k, b_k, a_{kt+1}) \leq p(b_k, a_{kt+1}) + W \right\} \geq 0.
\]

Note that \( i \) has the option to scrap \( j \) costlessly.
2 Equilibrium Properties

Theorem 1 For all \( b_j \) and all finite \( \alpha_j \), there exists \( A \) large enough such that
\[ V(u_{ij}, \alpha_j, b_j, A) = \beta W. \]

Proof. Conditional on \( b_j, \alpha_j, \) and \( a_{jt}, \)
\[ V(U, \alpha_j, b_j, a_{jt}) = U - c_{jt} + \beta \max [W, V(U, \alpha_j, b_j, a_{jt+1})] \tag{2} \]
because there is no other individual who will value \( j \) as much. Equation (2) satisfies
\[ V(U, \alpha_j, b_j, a_{jt}) \leq \max [W, U - c_{jt} + \frac{\beta}{1 - \beta} (U - \exp \{ \alpha_j + \lambda (b_j, a_{jt}) \})] . \]
Since \( U - c_{jt} \) is growing (negatively) without bound, there exists \( t \) large enough that satisfies the condition. But if the condition is satisfied for \( U \), then we should be able to show that it is satisfied for any \( u_{ij} < U \).

Theorem 2 For each pair \( (b_j, a_{jt}) \), \( \frac{\partial V(u_{ij}, \alpha_j, b_j, a_{jt})}{\partial \alpha_j} \leq 0 \) and \( \frac{\partial V(u_{ij}, \alpha_j, b_j, a_{jt})}{\partial a_{jt}} \leq 0 \), there exists an equilibrium price \( p(b_j, a_{jt}) \) with \( \frac{\partial p(b_j, a_{jt})}{\partial a_{jt}} \leq 0 \).

Proof. Conditional on \( b_j \) and \( \alpha_j \), let \( A \) be the first time such that \( V_i(u_{ij}, \alpha_j, b_j, a_{jt}) = W \). Then
\[ \frac{\partial V(u_{ij}, \alpha_j, b_j, A - 1)}{\partial \alpha_j} = -\frac{\partial c_{jt-1}}{\partial \alpha_j} < 0; \]
\[ \frac{\partial V(u_{ij}, \alpha_j, b_j, A - 1)}{\partial a_{jt-1}} = -\frac{\partial c_{jt-1}}{\partial a_{jt-1}} < 0. \]
Let \( g_\alpha(\alpha \mid b, a) \) be the density of \( \alpha \) and \( g_u(u \mid \alpha, b, a) \) be the conditional density of \( u \) given \( \alpha \) among owners of brand \( b \) cars of age \( a \). Let
\[ S(b, a) = \{(\alpha, u) : W \leq V(u, \alpha, b, a) \leq p + W \} \]
be the set of values of \((\alpha, u)\) where a seller wants to sell and
\[ B(b, a) = \{(\alpha, u, v, u') : (\alpha, v) \in S(b, a), \]
\[ E_\alpha[V(u, \alpha, b, a) - p(b, a) - W \mid (\alpha, v) \in S(b, a)] \]
\[ \geq \max_{j, a_j' > 0} E_{\alpha'}[V(u_j', \alpha_j', b_j', a_j') - p(b_j', a_j') - W \mid W \leq V(u_j', \alpha_j', b_j', a_j') \leq p(b_j', a_j') + W], \]
\[ E_\alpha[V(u, \alpha, b, a) - p(b, a) - W] \geq \max_j E_{\alpha'}[V(u_j', \alpha_j', b_j', 0) - p(b_j', 0) - W], \]
\[ E_\alpha[V(u, \alpha, b, a) - p(b, a) - W \mid (\alpha, v) \in S(b, a)] \geq 0 \]
be the set of values of \((\alpha, u, v, u')\) where a buyer wants to buy. Then, at age \(A\),
\(p(b_j, A)\) satisfies
\[
\gamma \int \left[ \int g_u(u | \alpha, b, a) g_\alpha(\alpha | b, a) \, d\alpha \right]^{-1} \int \cdots \int g_u(v | \alpha, b, a) g_\alpha(\alpha_j | b, a) g_u(u | \alpha, b, a) \cdot \
\prod_j \left[ g_u(u_j' | \alpha, b, a) g_\alpha(v_j' | \alpha, b, a) g_\alpha(\alpha_j' | b, a) \right] \, du' \, d\alpha
\]
where \(\gamma\) is the ratio of potential sellers to potential buyers. As \(p \to \infty\), the left hand side (LHS) of equation (3) approaches \(\gamma\) and the right hand side (RHS) approaches 0, and as \(p \to 0\), the LHS approaches \(0 \leq \zeta \leq \gamma\) and the RHS approaches \(\xi \geq 0\). Also, equation (3) is continuous in \(p\). If \(\zeta < \xi\), there exists \(p > 0\) that satisfies equation (3) with positive trade, and if \(\zeta \geq \xi\), \(p = 0\) maybe with some trade. If \(\zeta < \xi\), then, since \(\frac{\partial V(u_{i,j}, \alpha_j, b_j, A-1)}{\partial \alpha_j} < 0\), \(S(b, a)\) is increasing in \(a\) and \(B(b, a)\) is decreasing in \(a\). So \(p\) must be decreasing in \(a\).

3 Construction of Equilibrium Prices

Step 1: Pick an age \(A^*\) such that \(p(b_j, A^*) = 0\) for all \(b_j\).

Step 2: Look for a fixed point in \(R^{J+A^*}\). Standard iteration methods can be used.
4 Empirical Work

Assume we can get data on average price of a brand $b$ car at age $a$, $p(b,a)$ (from the “Blue Book”). Assume we can get average cost of maintainence of a brand $b$ car at age $a$,

$$\exp\left\{ \mu(b) + \frac{1}{2}\sigma^2 + \lambda(b,a) \right\}$$

(from equation (1)). Assume we can get data on the proportion of brand $b$ cars from vintage $t$ sold in year $s$,

$$\hat{\Psi}(s-t,b) = \frac{n_{stb}}{n_{ttb}}$$

where $n_{stb}$ is the number of brand $b$ automobiles of vintage $t$ sold in year $s$. The $plim\hat{\Psi}(s-t,b)$ is

$$\Psi(s-t,b) = \Pr[(u, \alpha) \in S(b,a)] = \iint_{S(b,a)} g_\alpha(u, \alpha, b, a) d\alpha$$

(maybe from DMV data). Equation (4) identifies $\mu(b)$ and $\lambda(b,a)$, and $p(b,a)$ is identified from “Blue Book” data. The remaining parameters to estimate are $\pi_2 = \left[ \sigma_u^2, \beta, U, \theta(b) \right]_{b=1}^B$. It is pretty likely that the DMV data overidentifies $\pi$.

We can estimate all of the parameters by first estimating $\pi_1 = \left\{ \{\mu(b), \lambda(b,a), p(b,a)\}_{a=1}^{A^*} \right\}_{b=1}$ using cost and blue book data and then estimating $\pi_2$ in a MLE routine using survivor data. The log likelihood function is

$$L = \sum_{b=1}^J L_b$$

with

$$L_b = \sum_{a=1}^{A^*} n_{ba} \log \left[ \iint_{S(b,a)} g_\alpha(u, \alpha, b, a) d\alpha \right].$$

5 Policy Issues

1) How much do equilibrium prices vary from full information equilibrium prices?

2) How much does the equilibrium survivor function vary from the full information survivor function?