Math Review

1 Rules of Differentiation

A. Use table as a guide

B. Basic rules

\[ \frac{d(ax^n)}{dx} = anx^{n-1} \]
\[ \frac{d \log x}{dx} = \frac{1}{x} \]
\[ \frac{de^{ax}}{dx} = ae^{ax} \]
\[ \frac{d}{dx} \sum_i a_i f_i(x) = \sum_i a_i \frac{df_i(x)}{dx} \]
\[ \frac{d}{dx} [f(x)g(x)] = f(x) \frac{dg(x)}{dx} + g(x) \frac{df(x)}{dx} \]
\[ \frac{d f(x)}{dx} g(x) = \frac{df(x)}{dx} g(x) - f(x) \frac{dg(x)}{dx} g(x) \]
\[ \frac{d}{dx} f(g(x)) = \frac{df}{dg} \frac{dg}{dx} \]

C. Chain rule

D. Meaning of derivative (draw picture)

E. Total derivative (draw topographic map with path to top of mountain).

Consider

\[ z = f(x, y) \]

Then

\[ dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \]
\[ = f_1 dx + f_2 dy \]
\[ \Rightarrow \frac{dz}{dx} = f_1 \frac{dx}{dx} + f_2 \frac{dy}{dx} \]
\[ = f_1 + f_2 \frac{dy}{dx} \]
F. Optimization

1) 1 variable: Consider a function $f(x)$ defined over $[a, b]$. A necessary condition for $x^*$ to maximize (minimize) $f(x)$ at an interior point ($a < x^* < b$) is

$$f'(x^*) = \frac{df(x^*)}{dx} = 0$$

and

$$f''(x^*) = \frac{d^2f(x^*)}{dx^2} > (<) 0.$$ 

Example:

$$f(x) = \exp\left\{-\frac{1}{2}x^2\right\}.$$ 

Then

$$f'(x) = -x \exp\left\{-\frac{1}{2}x^2\right\} \leq 0;$$

$$f''(x) = x^2 \exp\left\{-\frac{1}{2}x^2\right\} - \exp\left\{-\frac{1}{2}x^2\right\}$$

$$= \exp\left\{-\frac{1}{2}x^2\right\} [x^2 - 1] < 0.$$ 

Note that $f'(x) = 0$ only at $x = 0$ and, at $x = 0$, $f''(x) < 0$. Thus, $x = 0$ is potentially a maximum. Furthermore, since there are no other points where $f'(x) = 0$, $x = 0$ is a maximum, and there are no minima. Draw a picture to confirm.

A necessary condition for $x^*$ to maximize (minimize) $f(x)$ at a corner ($x^* = a$ or $x^* = b$) is

$$f'(a) \leq (\geq) 0;$$

$$f'(b) \geq (\leq) 0.$$ 

Example: Consider

$$f(x) = (x - 5)^2$$

for $-100 \leq x \leq 50$. Then

$$f'(x) = 2(x - 5);$$

$$f''(x) = 2.$$ 

Note that $f'(x) = 0$ only at $x = 5$, and, at $x = 5$, $f''(x) = 2 > 0$. Thus, $x = 5$ is a unique minimum. At $x = -100$, $f'(x) = -210 < 0$. Thus $x = -100$ is a
potential maximum. At $x = 50$, $f'(x) = 90 > 0$. Thus, $x = 50$ is a potential maximum. But, at $x = -100$, $f(x) = 105^2$, and, at $x = 50$, $f(x) = 45^2 < 105^2$. Thus, only $x = -100$ is a maximum.

2) Multiple variables: Consider $f(x, y, z)$. A necessary condition for $(x^*, y^*, z^*)$ to maximize (minimize) $f(x, y, z)$ at an interior point is

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0;$$

$$\begin{bmatrix}
\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\
\frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\
\frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2}
\end{bmatrix}$$

is negative (positive) semidefinite.

G. Implicit derivatives: Consider the equation

$$f(x, y) = c.$$  

Then the total derivative of the equation is

$$f_1 dx + f_2 dy = 0.$$  

We can solve for

$$\frac{dy}{dx} = -\frac{f_1}{f_2}.$$  

Example: Consider the equation for a circle:

$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$  

The total derivative is

$$2(x - x_0) \, dx + 2(y - y_0) \, dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(x - x_0)}{(y - y_0)}.$$  

H. Change of parameter at an optimum: Consider a function $f(x, y)$, and, for a given $y$, define $x^*(y)$ as the value of $x$ that maximizes (or minimizes) $f(x, y)$. Then

$$\frac{df}{dy} (x^*(y), y) = \frac{\partial f}{\partial y} (x^*(y), y) + \frac{\partial f}{\partial x} (x^*(y), y) \frac{dx}{dy} (y).$$  

But, since

$$\frac{\partial f}{\partial x} (x^*(y), y) = 0$$

at $x^*(y)$ (given the necessary condition for an optimum),

$$\frac{df}{dy} (x^*(y), y) = \frac{\partial f}{\partial y} (x^*(y), y).$$

This is called the Envelope Theorem.