Math Review

1 Rules of Differentiation

A. Use table as a guide

B. Basic rules

$$\frac{d\left(ax^n\right)}{dx} = anx^{n-1}$$

$$\frac{d\log x}{dx} = \frac{1}{x}$$

$$\frac{de^{ax}}{dx} = ae^{ax}$$

$$\frac{d}{dx}\sum_i a_i f_i\left(x\right) = \sum_i a_i \frac{df_i\left(x\right)}{dx}$$

$$\frac{d}{dx}\left[f\left(x\right)g\left(x\right)\right] = f\left(x\right)\frac{dg\left(x\right)}{dx} + g\left(x\right)\frac{df\left(x\right)}{dx}$$

$$\begin{aligned}\frac{d}{dx}\frac{f\left(x\right)}{g\left(x\right)} &= \frac{\frac{df(x)}{dx}}{g\left(x\right)} - \frac{f\left(x\right)\frac{dg(x)}{dx}}{g\left(x\right)^2} \\ &= \frac{g\left(x\right)\frac{df(x)}{dx} - f\left(x\right)\frac{dg(x)}{dx}}{g\left(x\right)^2}\end{aligned}$$

C. Chain rule

$$\frac{d}{dx}f\left[g\left(x\right)\right] = \frac{df}{dg}\frac{dg}{dx}$$

D. Meaning of derivative (draw picture)

E. Total derivative (draw topographic map with path to top of mountain). Consider

$$z = f\left(x, y\right)$$

Then

$$\begin{aligned} dz &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\ &= f_1 dx + f_2 dy \\ \Rightarrow & \frac{dz}{dx} = f_1\frac{dx}{dx} + f_2\frac{dy}{dx} \\ &= f_1 + f_2\frac{dy}{dx}\end{aligned}$$

F. Optimization

1) 1 variable: Consider a function $f(x)$ defined over $[a,b]$. A necessary condition for x^* to maximize (minimize) $f(x)$ at an interior point $(a < x^* < b)$ is

$$f^{'}(x^*) = \frac{df(x^*)}{dx} = 0$$

and

$$f"(x^*) = \frac{d^2 f(x^*)}{dx^2} > (<)\, 0.$$

Example:

$$f(x) = \exp\left\{-\frac{1}{2}x^2\right\}.$$

Then

$$\begin{aligned} f^{'}(x) &= -x\exp\left\{-\frac{1}{2}x^2\right\} \leq 0; \\ f"(x) &= x^2\exp\left\{-\frac{1}{2}x^2\right\} - \exp\left\{-\frac{1}{2}x^2\right\} \\ &= \exp\left\{-\frac{1}{2}x^2\right\}\left[x^2-1\right] < 0. \end{aligned}$$

Note that $f^{'}(x) = 0$ only at $x = 0$ and, at $x = 0$, $f"(x) < 0$. Thus, $x = 0$ is potentially a maximum. Furthermore, since there are no other points where $f^{'}(x) = 0$, $x = 0$ is a maximum, and there are no minima. Draw a picture to confirm.

A necessary condition for x^* to maximize (minimize) $f(x)$ at a corner ($x^* = a$ or $x^* = b$) is

$$\begin{aligned} f^{'}(a) &\leq (\geq)\, 0; \\ f^{'}(b) &\geq (\leq)\, 0. \end{aligned}$$

Example: Consider

$$f(x) = (x-5)^2$$

for $-100 \leq x \leq 50$. Then

$$\begin{aligned} f^{'}(x) &= 2(x-5); \\ f^{"}(x) &= 2. \end{aligned}$$

Note that $f^{'}(x) = 0$ only at $x = 5$, and, at $x = 5$, $f^{"}(x) = 2 > 0$. Thus, $x = 5$ is a unique minimum. At $x = -100$, $f^{'}(x) = -210 < 0$. Thus $x = -100$ is a potential maximum. At $x = 50$, $f^{'}(x) = 90 > 0$. Thus, $x = 50$ is a potential maximum. But, at $x = -100$, $f(x) = 105^2$, and, at $x = 50$, $f(x) = 45^2 < 105^2$. Thus, only $x = -100$ is a maximum.

2) Multiple variables: Consider $f(x, y, z)$. A necessary condition for (x^*, y^*, z^*) to maximize (minimize) $f(x, y, z)$ at an interior point is

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0;$$

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}$$

is negative (positive) semidefinite.

G. Implicit derivatives: Consider the equation

$$f(x, y) = c.$$

Then the total derivative of the equation is

$$f_1 dx + f_2 dy = 0.$$

We can solve for

$$\frac{dy}{dx} = -\frac{f_1}{f_2}.$$

Example: Consider the equation for a circle:

$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

The total derivative is

$$2(x - x_0)\,dx + 2(y - y_0)\,dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(x - x_0)}{(y - y_0)}.$$

H. Change of parameter at an optimum: Consider a function $f(x, y)$, and, for a given y, define $x^*(y)$ as the value of x that maximizes (or minimizes) $f(x, y)$. Then

$$\frac{df(x^*(y), y)}{dy} = \frac{\partial f(x^*(y), y)}{\partial y} + \frac{\partial f(x^*(y), y)}{\partial x}\frac{\partial x^*(y)}{\partial y}.$$

But, since

$$\frac{\partial f(x^*(y), y)}{\partial x} = 0$$

at $x^*(y)$ (given the necessary condition for an optimum),

$$\frac{df(x^*(y), y)}{dy} = \frac{\partial f(x^*(y), y)}{\partial y}.$$

This is called the Envelope Theorem.