Asymptotics

1 Example

\[ X_i \sim iid(\mu, \sigma^2), \quad i = 1, 2, \ldots, n \]

\[ \bar{X} = \frac{1}{n} \sum_i X_i \]

\[ \Rightarrow E\bar{X} = E\frac{1}{n} \sum_i X_i = \frac{1}{n} \sum_i EX_i = \frac{1}{n} \sum_i \mu = \mu \]

\[ Var\bar{X} = Var\left[\frac{1}{n} \sum_i X_i\right] = \frac{1}{n^2} \sum_i VarX_i = \frac{1}{n^2} \sum_i \sigma^2 = \frac{\sigma^2}{n} \]

\[ Var\bar{X} \to 0 \text{ as } n \to \infty \]
\[ \bar{X} \to \mu \text{ as } n \to \infty. \]

2 Concepts

Probability Limits: Let \( S_n \) be a statistic whose properties depend on \( n \). Then (weak consistency)

\[ plim S_n = \theta \iff \lim_{n \to \infty} Pr[|S_n - \theta| < \varepsilon] = 1 \quad \forall \varepsilon > 0. \]

We say that \( S_n \) is a weakly consistent estimator of \( \theta \).

Example (continued):

\[ plim \bar{X}_n = \mu? \]

\[ Pr\left[|\bar{X}_n - \mu| < \varepsilon\right] = Pr\left[-\varepsilon < \bar{X}_n - \mu < \varepsilon\right] \]

\[ = \Pr\left[\frac{-\sqrt{n}\varepsilon}{\sigma} < \frac{\sqrt{n}}{\sigma}(\bar{X}_n - \mu) < \frac{\sqrt{n}\varepsilon}{\sigma}\right] \]

\[ = \Pr\left[\frac{-\sqrt{n}\varepsilon}{\sigma} < Z < \frac{\sqrt{n}\varepsilon}{\sigma}\right] \]

where \( Z \sim N(0, 1) \). As \( n \to \infty \), \( \frac{\sqrt{n}\varepsilon}{\sigma} \to \infty \) for all fixed \( \varepsilon \Rightarrow \)

\[ \lim_{n \to \infty} Pr\left[\frac{-\sqrt{n}\varepsilon}{\sigma} < Z < \frac{\sqrt{n}\varepsilon}{\sigma}\right] = 1 \]

\[ \Rightarrow plim \bar{X}_n = \mu. \]
Alternatively, we say that, if
\[
Pr \left[ \lim_{n \to \infty} |S_n - \theta| < \varepsilon \right] = 1,
\]
then \(S_n\) is a strongly consistent estimator of \(\theta\).

Example (continued):
\[
|\overline{X}_n - \mu| = \frac{\sigma}{\sqrt{n}} \left| \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \right| = \frac{\sigma}{\sqrt{n}} |Z|
\]

where \(Z \sim N(0,1)\) ⇒
\[
\lim_{n \to \infty} \frac{\sigma}{\sqrt{n}} |Z| = 0
\]
\[
\Rightarrow \Pr \left[ \lim_{n \to \infty} |\overline{X}_n - \mu| < \varepsilon \right] = 1 \forall \varepsilon > 0.
\]

Thus, \(\overline{X}_n\) is a strongly consistent estimator of \(\mu\).

New example:
\[
S_n = \begin{cases} 
\overline{X}_n & \text{with probability } 1 - \frac{1}{n} \\
n & \text{with probability } \frac{1}{n}
\end{cases}
\]

Note that
\[
\Pr [|S_n - \mu| < \varepsilon] = \begin{cases} 
\Pr \left[ |Z| < \frac{\sqrt{\varepsilon}}{\sigma} \right] & \text{with probability } 1 - \frac{1}{n} \\
0 & \text{with probability } \frac{1}{n}
\end{cases}
\]

and that, as \(n \to \infty\), \(\frac{1}{n} \to 0\) and the second term disappears. ⇒ \(S_n\) is a weakly consistent estimator of \(\mu\). However,
\[
\lim_{n \to \infty} |S_n - \mu|
\]
does not exist. So \(S_n\) is not a strongly consistent estimator of \(\mu\). Explain why this example is relevant.

3 Properties

\(\text{plim } c = c\)

\(\text{plim } cX_n = c \text{plim } X_n\)

\(\text{plim } (X_n + Y_n) = \text{plim} X_n + \text{plim} Y_n\)

\(\text{plim } (X_n Y_n) = (\text{plim} X_n) (\text{plim} Y_n)\)

\(\text{plim } g(X_{1n}, X_{2n}, \ldots, X_{mn}) = g(\text{plim} X_{1n}, \text{plim} X_{2n}, \ldots, \text{plim} X_{mn})\)

Compare consistency and unbiasedness: If \(EX_n \to \mu\) and \(Var X_n \to 0\), then \(\text{plim } X_n = \mu\). The converse is not true. Examples:
1. 

\[ X_i \sim iid (\mu, \sigma^2) \]
\[ S_n = \frac{1}{m} \sum_{i=1}^{m} X_i \]

for fixed \( m \). Then

\[ ES_n = \mu, \]
\[ VarS_n = \frac{\sigma^2}{m} \to 0 \text{ as } n \to \infty \]

\( \Rightarrow plim S_n \) does not exist.

2. 

\[ X_i \sim iid (\mu, \sigma^2) \]
\[ S_n = \left[ \frac{1}{n} \sum_{i=1}^{m} X_i \right]^{-1} \]

Then

\[ plimS_n = plim \left[ \frac{1}{n} \sum_{i=1}^{m} X_i \right]^{-1} \]
\[ plim \left[ \frac{1}{n} \sum_{i=1}^{m} X_i \right]^{-1} = \frac{1}{\mu} \]

But

\[ ES_n = E(\bar{X}_n)^{-1} \neq (E\bar{X}_n)^{-1} \]

and, in fact, for many cases does not exist.

4 Central Limit Theorem

Let

\[ X_i \sim iid (\mu, \sigma^2), \ i = 1, 2, ..., n. \]

Then

\[ \sqrt{n}(\bar{X} - \mu) / \sigma \sim N(0, 1) \]

for a large class of distributions for \( X_i \). The CLT generalizes in many ways, the most important being to allow for heterogeneity in \( X_i \).
One more example: let \( U \sim \chi^2_m \), \( V \sim \chi^2_n \) with \( U, V \) independent. Then

\[
Z = \frac{U/m}{V/n} \sim F_{m,n}.
\]

What happens as \( n \to \infty \)? Define

\[
V = \sum_{i=1}^{n} W_i
\]

where \( W_i \sim iid \chi^2_1 \)

\[
\Rightarrow EW_i = 1, VarW_i = 2
\]

\[
\Rightarrow \text{plim} \frac{V}{n} = 1
\]

\[
\Rightarrow Z = \frac{U/m}{V/n} \to \frac{U}{m} \text{ as } n \to \infty
\]

\[
\Rightarrow mZ \to U \sim \chi^2_m.
\]