Asymptotics

1 Example

$$
\begin{aligned}
X_i &\sim iid\left(\mu,\sigma^2\right), \quad i=1,2,..,n \\
\overline{X} &= \frac{1}{n}\sum_i X_i \\
&\Rightarrow E\overline{X}=E\frac{1}{n}\sum_i X_i=\frac{1}{n}\sum_i EX_i=\frac{1}{n}\sum_i \mu=\mu \\
Var\overline{X} &= Var\left[\frac{1}{n}\sum_i X_i\right]=\frac{1}{n^2}\sum_i VarX_i=\frac{1}{n^2}\sum_i\sigma^2=\frac{\sigma^2}{n}.
\end{aligned}
$$

$$
\begin{aligned}
Var\overline{X} &\rightarrow 0 \text{ as } n\rightarrow\infty \\
\overline{X} &\rightarrow \mu \text{ as } n\rightarrow\infty.
\end{aligned}
$$

2 Concepts

Probability Limits: Let S_n be a statistic whose properties depend on n. Then (weak consistency)

$$plimS_n=\theta \text{ iff } \lim_{n\rightarrow\infty}\Pr\left[|S_n-\theta|<\varepsilon\right]=1\ \forall\varepsilon>0.$$

We say that S_n is a weakly consistent estimator of θ.

Example (continued):

$$plim\overline{X}_n=\mu?$$

$$
\begin{aligned}
\Pr\left[\left|\overline{X}_n-\mu\right|<\varepsilon\right] &= \Pr\left[-\varepsilon<\overline{X}_n-\mu<\varepsilon\right] \\
&= \Pr\left[\frac{-\sqrt{n}\varepsilon}{\sigma}<\frac{\sqrt{n}\left(\overline{X}_n-\mu\right)}{\sigma}<\frac{\sqrt{n}\varepsilon}{\sigma}\right] \\
&= \Pr\left[\frac{-\sqrt{n}\varepsilon}{\sigma}<Z<\frac{\sqrt{n}\varepsilon}{\sigma}\right]
\end{aligned}
$$

where $Z\sim N(0,1)$. As $n\rightarrow\infty$, $\frac{\sqrt{n}\varepsilon}{\sigma}\rightarrow\infty$ for all fixed $\varepsilon\Rightarrow$

$$
\begin{aligned}
\lim_{n\rightarrow\infty}\Pr\left[\frac{-\sqrt{n}\varepsilon}{\sigma}<Z<\frac{\sqrt{n}\varepsilon}{\sigma}\right] &= 1 \\
&\Rightarrow plim\overline{X}_n=\mu.
\end{aligned}
$$

Alternatively, we say that, if

$$\Pr\left[\lim_{n\to\infty}|S_n-\theta|<\varepsilon\right]=1,$$

then S_n is a strongly consistent estimator of θ.

Example (continued):

$$\left|\overline{X}_n-\mu\right|=\frac{\sigma}{\sqrt{n}}\left|\frac{\overline{X}_n-\mu}{\sigma/\sqrt{n}}\right|=\frac{\sigma}{\sqrt{n}}|Z|$$

where $Z\sim N(0,1)\Rightarrow$

$$\begin{aligned}\lim_{n\to\infty}\frac{\sigma}{\sqrt{n}}|Z| &= 0\\ &\Rightarrow \Pr\left[\lim_{n\to\infty}\left|\overline{X}_n-\mu\right|<\varepsilon\right]=1\ \forall\varepsilon>0.\end{aligned}$$

Thus, $\overline{X}_n$ is a strongly consistent estimator of μ.

New example:

$$S_n=\begin{cases}\overline{X}_n & \text{with probability } 1-\frac{1}{n}\\ n & \text{with probability } \frac{1}{n}\end{cases}.$$

Note that

$$\Pr\left[|S_n-\mu|<\varepsilon\right]=\begin{cases}\Pr\left[|Z|<\frac{\sqrt{n}\varepsilon}{\sigma}\right] & \text{with probability } 1-\frac{1}{n}\\ 0 & \text{with probability } \frac{1}{n}\end{cases}$$

and that, as $n\to\infty$, $\frac{1}{n}\to0$ and the second term disappears. $\Rightarrow S_n$ is a weakly consistent estimator of μ. However,

$$\lim_{n\to\infty}|S_n-\mu|$$

does not exist. So S_n is not a strongly consistent estimator of μ. Explain why this example is relevant.

3 Properties

$$plim\ c=c$$

$$plim\ cX_n=c plim\ X_n$$

$$plim\left(X_n+Y_n\right)=plim X_n+plim Y_n$$

$$plim\left(X_nY_n\right)=\left(plim X_n\right)\left(plim Y_n\right)$$

$$plim\ g\left(X_{1n},X_{2n},..,X_{mn}\right)=g\left(plim X_{1n},plim X_{2n},..,plim X_{mn}\right)$$

Compare consistency and unbiasedness: If $EX_n\to\mu$ and $VarX_n\to0$, then $plim\ X_n=\mu$. The converse is not true. Examples:

1.

$$\begin{aligned} X_i &\sim iid\left(\mu, \sigma^2\right) \\ S_n &= \frac{1}{m}\sum_{i=1}^{m} X_i \end{aligned}$$

for fixed m. Then

$$\begin{aligned} ES_n &= \mu, \\ VarS_n &= \frac{\sigma^2}{m} \nrightarrow 0 \text{ as } n \to \infty \end{aligned}$$

$\Rightarrow plim\ S_n$ does not exist.

2.

$$\begin{aligned} X_i &\sim iid\left(\mu, \sigma^2\right) \\ S_n &= \left[\frac{1}{n}\sum_{i=1}^{m} X_i\right]^{-1}. \end{aligned}$$

Then

$$\begin{aligned} plimS_n &= plim\left[\frac{1}{n}\sum_{i=1}^{m} X_i\right]^{-1} \\ \left[plim\ \overline{X}_n\right]^{-1} &= \frac{1}{\mu}. \end{aligned}$$

But

$$ES_n = E\left(\overline{X}_n\right)^{-1} \neq \left(E\overline{X}_n\right)^{-1}$$

and, in fact, for many cases does not exist.

4 Central Limit Theorem

Let

$$X_i \sim iid\left(\mu, \sigma^2\right), \quad i = 1, 2, .., n.$$

Then

$$\sqrt{n}\left(\overline{X} - \mu\right)/\sigma \sim N(0, 1)$$

for a large class of distributions for X_i. The CLT generalizes in many ways, the most important being to allow for heterogeneity in X_i.

One more example: let $U \sim \chi^2_m$, $V \sim \chi^2_n$ with U, V independent. Then

$$Z = \frac{U/m}{V/n} \sim F_{m,n}.$$

What happens as $n \to \infty$? Define

$$V = \sum_{i=1}^{n} W_i$$

where $W_i \sim iid\chi^2_1$

$$\begin{aligned}
&\Rightarrow \quad EW_i = 1, VarW_i = 2 \\
&\Rightarrow \quad plim\ \frac{V}{n} = 1 \\
&\Rightarrow \quad Z = \frac{U/m}{V/n} \to \frac{U}{m} \text{ as } n \to \infty \\
&\Rightarrow \quad mZ \to U \sim \chi^2_m.
\end{aligned}$$