1 Do 9 out 10 questions. (90 points)

1. Let $X_i \sim iid F(\cdot), i = 1, 2, \ldots, n$, where
   \[ F(x) = \exp \{-\lambda x\}, x \leq 0. \]
   Using a Wald test, test
   \[ H_0 : \lambda = 2 \text{ vs } H_A : \lambda \neq 2. \]

2. Let $X_i \sim iid F(\cdot), i = 1, 2, \ldots, n$, where
   \[ F(x) = 1 - \exp \{-\lambda (x - 1)\}, x \geq 1, \]
   and let
   \[ \pi(\lambda) = ae^{-\alpha \lambda} \]
   be the prior density for $\lambda$. Find the posterior density for $\lambda$ after observing
   \[ \{X_i\}_{i=1}^{n}. \]

3. Prove that the first derivative of a moment generating function, evaluated
   at 0, provides the first moment of the corresponding distribution.

4. Define $X' = (X_1, X_2, \ldots, X_K)$, and let
   \[ f(x) = \prod_{k=1}^{K} 1(0 \leq x_k \leq 1). \]
   Find
   \[ \Pr[X_2 + X_3 < X_4 \mid X_5]. \]

5. Consider the model,
   \[ Y_i = g(X_i), \]
   \[ X_i \sim iid F(\cdot). \]
   Consider estimating $g(\cdot)$ as
   \[ \hat{g}(x) = \sum_{i=1}^{n} y_i K\left(\frac{x_i - x}{b}\right) / \sum_{i=1}^{n} K\left(\frac{x_i - x}{b}\right). \]
   Derive $E\hat{g}(x)$. 
6. Let \( X \sim \lambda_x \exp \{-\lambda_x x\} \) and \( Y \sim \lambda_y \exp \{-\lambda_y y\} \). Find the density of \( Z = X - Y \).

7. Let \( X_i \sim iidU(0, \theta) \), and define \( \overline{X} = n^{-1} \sum_{i=1}^{n} X_i \). Explain how the fact that the support of \( X_i \) is bounded affects the application of the Central Limit Theorem to the asymptotic distribution of \( \sqrt{n} (\overline{X} - \text{plim}X) \).

8. Let \( X_i \sim \text{indPoisson}(\lambda_i), \ i = 1, 2, \ldots, n \), where

\[
\log \lambda_i = \alpha + \beta z_i.
\]

Given data, \( \{x_i, z_i\}_{i=1}^{n} \), derive the MLE of \( \theta = (\alpha, \beta) \) and show that it is consistent.

9. Let \( X_i \sim \text{indPoisson}(\lambda_i), \ i = 1, 2, \ldots, n \), where

\[
\log \lambda_i = \alpha + \beta z_i.
\]

Given data, \( \{x_i, z_i\}_{i=1}^{n} \), derive the MOM estimator of \( \theta = (\alpha, \beta) \) and show that it is consistent.

10. Let \( X_i \sim iidN(0, 1), \ i = 1, 2, \ldots, m \), and let \( Z_i \sim iidN(0, 1), \ i = 1, 2, \ldots, n \). Define

\[
Y = \frac{m^{-1} \sum_{i=1}^{m} X_i^2}{n^{-1} \sum_{i=1}^{n} Z_i^2}.
\]

Find the distribution of \( mY \) as \( n \to \infty \).

### 2. Do 3 out of 4 question. (90 points)

1. Let \( X_i^* = (X_{i1}, X_{i2}) \), and assume that \( X_i \sim iidF(\cdot), \ i = 1, 2, \ldots, n \). Define \( Z_{ikm} = 1(\tau_{1k} \leq X_{i1} < \tau_{1k+1}, \tau_{2m} \leq X_{i2} < \tau_{2m+1}) \). \([\text{In words, divide } \mathbb{R}^2 \text{ into } KM \text{ rectangles with nodes at } (\tau_{1k}, \tau_{2m}), (\tau_{1k+1}, \tau_{2m}), (\tau_{1k}, \tau_{2m+1}), \text{ and } (\tau_{1k+1}, \tau_{2m+1}) \text{ for a representative rectangle, } k = 1, 2, \ldots, K \text{ and } m = 1, 2, \ldots, M, \text{ and let } Z_{ikm} \text{ be a binary indicator of whether } X_i \text{ is in the particular representative rectangle}.\] Define \( Z_i \) as the vector of \( Z_{ikm} \)'s for all \( k = 1, 2, \ldots, K \text{ and } m = 1, 2, \ldots, M \). Given data, \( \{z_i\}_{i=1}^{n} \), suggest how to test if \( F(\cdot) \) is bivariate normal.

2. Let \( X_i^* \sim iidN(\mu, 1) \), and define \( X_i = 1(X_i^* > c) \). Consider a dataset, \( \{x_i\}_{i=1}^{n} \), and consider the MLE of \( \theta = (\mu, c) \).
   a) Explain why one can not estimate \( \theta \) using MLE.
   b) Consider restricting \( c = 0 \). Then

\[
\log L = \sum_{i} x_i \log \Phi(\mu) + (1 - x_i) \log \Phi(-\mu).
\]

For values of \( \mu : |\mu| > 5 \), either \( \Phi(\mu) \) or \( \Phi(-\mu) \) is too close to zero to take a log on a computer with limited precision. Using L'Hopital’s Rule, suggest how to approximate \( \log \Phi(\mu) \), \( \log \Phi(-\mu) \) for \( |\mu| > 5 \).
3. Consider the history of earthquakes in Japan. Starting at some specified time, let $T_i$ be the amount of time between the $(i - 1)$th and $i$th earthquake, and let $D_i$ be the amount of damage done by the $i$th earthquake. Construct a flexible but parsimonious specification for the joint density of $\{T_i, D_i\}_{i=1}^n$. Your model should be able to explain aftershocks and allow for the possibility that timing and damage are dependent on each other.

4. Let $Y_{ijkt}$ be a binary indicator for whether the $k$th family member in family $j$ of city $i$ has a particular acute disease at time $t$. Construct a model for $Y_{ijkt}$ that accounts for contagiousness within families and, to a lesser degree, within cities. Your model should also allow for improvements over time in disease containment. Finally, it should allow for infrequent occurrences of epidemics.