The trace formula for a representation of the fundamental group of a surface in \( \text{PSL}(2, \mathbb{C}) \)

Sara Maloni

April 25, 2012
Let $\gamma$ be a simple closed curve on the surface $\Sigma$ and let $i(\gamma) = (m_1, t_1, \cdots, m_\xi, t_\xi)$ be the Dehn-Thurston coordinates of $\gamma$ w.r.t. $\mathcal{P}$ where $\sum_{i=1}^\xi m_i \neq 0$. Let $\rho : \Pi_1(\Sigma) \to \text{PSL}(2, \mathbb{C})$ be the holonomy representation corresponding to glueing around horocycles with parameters $\tau_1, \cdots, \tau_\xi$ and let $\gamma$ be represented by $g \in \Pi_1(\Sigma)$. Then $\text{Tr} \rho(g)$ is a polynomial in $\tau_1, \cdots, \tau_\xi$ such that:

$$\text{Tr} \rho(g) = \pm i^m 2^h (\tau_1 - \frac{(t_1 - m_1)}{m_1})^{m_1} \cdots (\tau_\xi - \frac{(t_\xi - m_\xi)}{m_\xi})^{m_\xi} + O(m - 2)$$

where $m = \sum_{i=1}^\xi m_i$, $h = \#\{\text{archetypes in } \gamma\}$ and where $O(m - 2)$ means terms of degree at most $m_i$ in the variable $\tau_i$, but with total degree in $\tau_1 \cdots \tau_\xi$ at most $m - 2$. 

Sara Maloni

The trace formula for a representation of the fundamental group of a surface in $\text{PSL}(2, \mathbb{C})$. 

Sara Maloni
1 Introduction
   - Dehn-Thurston coordinates
   - Plumbing parameters

2 Our recipe
   - Complex projective structure
   - Overlap maps in our case
   - Calculations

3 Trace formula
   - Example: the four holed sphere
   - Trace formula
Brief historic excursus about Dehn-Thurston coordinates

- 1922 Max Dehn in the Lecture notes from Braslau.
- 1976 William Thurston in “On the geometry and dynamics of diffeomorphisms of surfaces”.
- 1982 Robert Penner in “The action of the mapping class group on curves in surfaces”.
- 2008 Dylan Thurston in “Geometric intersection of curves on surfaces”. 
The first elementary transformation is given by the following formulas:

\[
\begin{align*}
\lambda'_{11} &= \{ r - |t_1| \} \lor 0 \\
\lambda'_{12} &= \lambda_{13} = L + \lambda_{11} - \lambda'_{11} \\
\lambda'_{23} &= |t_1| + \lambda'_{11} - L \\
t'_2 &= t_2 + \lambda_{11} + \{(L - \lambda'_{11}) \land t_1\} \lor 0 \\
t'_1 &= -\text{sgn}(t_1) \{ \lambda_{23} + L - \lambda'_{11} \},
\end{align*}
\]

where \( L = \lambda_{12} = \lambda_{13} \) and \( \text{sgn}(0) \) is taken to be \(-1\).

The second elementary transformation is given by the following formulas:

\[
\begin{align*}
\kappa'_{11} &= \kappa_{22} + \lambda_{33} + \{(L - \kappa_{13}) \lor 0 + \{-L - \lambda_{12}\} \lor 0 \\
\kappa'_{22} &= \{ L \lor \lambda_{11} \lor (\kappa_{13} - \lambda_{12} - L) \} \lor 0 \\
\kappa'_{33} &= \{- L \lor \kappa_{11} \lor (\lambda_{12} - \kappa_{13} + L) \} \lor 0 \\
\kappa'_{23} &= \{ \kappa_{13} \lor \lambda_{12} \lor (\kappa_{13} - L) \lor (\lambda_{12} + L) \} \lor 0 \\
\kappa'_{12} &= -2\kappa'_{22} - \kappa'_{23} + \kappa_{13} + \kappa_{23} + 2\kappa_{33} \\
\kappa'_{13} &= -2\kappa'_{33} - \kappa_{23} + \lambda_{12} + \lambda_{33} + 2\lambda_{22} \\
\lambda'_{11} &= \lambda_{23} + \lambda_{33} + \{ K - \lambda_{13} \} \lor 0 + \{-K - \kappa_{12}\} \lor 0 \\
\lambda'_{22} &= \{ K \lor \kappa_{11} \lor (\lambda_{13} - \kappa_{12} - K) \} \lor 0 \\
\lambda'_{33} &= \{-K \lor \lambda_{11} \lor (\kappa_{12} - \lambda_{13} + K) \} \lor 0 \\
\lambda'_{23} &= \{ \lambda_{13} \lor \kappa_{12} \lor (\lambda_{13} - K) \lor (\kappa_{12} + K) \} \lor 0 \\
\lambda'_{12} &= -2\lambda'_{22} - \lambda'_{23} + \lambda_{13} + \lambda_{23} + 2\lambda_{33} \\
\lambda'_{13} &= -2\lambda'_{33} - \lambda'_{23} + \kappa_{12} + \kappa_{23} + 2\kappa_{22} \\
t'_2 &= t_2 + \lambda_{33} + \{(\lambda_{13} - \lambda'_{23} - 2\lambda'_{22}) \lor (K + \lambda'_{33} - \lambda'_{22}) \} \lor 0 \\
t'_3 &= t_3 - \lambda'_{33} + \{(L + \kappa_{33} - \kappa_{22}) \lor (\kappa_{23} + 2\kappa'_{33} - \lambda_{12}) \} \lor 0 \\
t'_4 &= t_4 - \lambda'_{33} + \{(K + \lambda'_{33} - \lambda'_{22}) \lor (\lambda_{23} + 2\lambda'_{33} - \lambda_{12}) \} \lor 0 \\
t'_5 &= t_5 + \lambda_{33} + \{(\kappa_{13} - \kappa'_{23} - 2\kappa_{22}) \lor (L + \kappa_{33} - \kappa_{22}) \} \lor 0 \\
t'_1 &= \kappa_{22} + \lambda_{23} + \kappa_{33} + \lambda_{33} - \{\lambda_{11} + \kappa'_{11} + (t'_2 - t_2) + (t'_5 - t_5)\} \\
    &\quad + \left[ \text{sgn}(L + K + \lambda'_{33} - \lambda'_{22} + \kappa'_{33} - \kappa'_{22}) \right] (t_1 + \lambda_{33} + \kappa'_{33}),
\end{align*}
\]

where \( L = \lambda_{11} + t_2, \) \( K = \kappa_{11} + t_1 \) and

\[
\text{sgn}(0) = \begin{cases} 
+1, & \text{if } \lambda_{12} - 2\kappa'_{33} - \kappa'_{23} \neq 0; \\
-1, & \text{otherwise}.
\end{cases}
\]

**Figure:** Penner’s formulae
Dylan Thurston’s elementary moves

**Lemma 25.** With the setup as above, $mF$ is given by

$$mF = \max(mA + mC - mE, mB + mD - mF, |tE| + g(mA, mB; mE) + g(mC, mD; mE))$$

where

$$g(x, y; z) = \max(0, x - z, y - z, (x + y - z)/2).$$

**Lemma 27.** With the setup as above, the twist of $F$ is given by

$$tF = -tE$$

if $mE + mF = (mA + mC) \lor (mB + mD)$, and

$$tF = -\text{sign}(tE)(mF - g(mA, mD; mF) - g(mB, mC; mF))$$

otherwise, where $g$ is as in Lemma 25.

**Figure:** Dylan Thurston’s formulae
Fix a **pants decomposition** $\mathcal{P} = \{\gamma_1, \ldots, \gamma_\xi\}$ and a **marking** on it. The marking could be given in three ways: fixing

- a **reversing map** $R : \Sigma \longrightarrow \Sigma$;
- an **hexagonal decomposition** $H$ which meets each pants curve twice;
- **dual curves** $D_i$ such that $i(D_i, \gamma_j) = 2\delta_{ij}$.

Given a multi-curve $\alpha$, for each pants curve $\gamma_i$ we will define:

1. the **length parameter** $m_i(\alpha) = i(\alpha, \gamma_i)$;
2. the **twist parameter** $t_i(\alpha) = \hat{i}(\alpha \cap A_i, D_i \cap A_i)$.
Dehn-Thurston coordinates

Fix a **pants decomposition** \( \mathcal{P} = \{\gamma_1, \ldots, \gamma_\xi\} \) and a **marking** on it. The marking could be given in three ways: fixing

- a *reversing map* \( R : \Sigma \to \Sigma \);
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Given a multi-curve \( \alpha \), for each pants curve \( \gamma_i \) we will define:

1. the **length parameter** \( m_i(\alpha) = i(\alpha, \gamma_i) \);
2. the **twist parameter** \( t_i(\alpha) = \hat{i}(\alpha \cap A_i, D_i \cap A_i) \).

**Theorem (Dehn’s theorem)**

*There is a parametrisation of \( SC(\Sigma) \) by a subset of \( (\mathbb{Z}_{\geq 0})^\xi \times \mathbb{Z}^\xi \).*
Twisting number

The literature doesn’t agree about the definition of the twisting number. In fact, we have the different definitions given by:

- A. Fathi and V. Poénaru in “Travaux de Thurston sur les surfaces” (denoted $t'_i$);
- Feng Luo (using a multiplicative structure on $SC(\Sigma)$) (denoted $t''_i$);
- Robert Penner (using the train tracks) (denoted $p_i$);
- Dylan Thurston (using the algebraic intersection number) (denoted $t_i$).
Twisting number

The literature doesn’t agree about the definition of the twisting number. In fact, we have the different definitions given by:

- A. Fathi and V. Poénaru in “Travaux de Thurston sur les surfaces” (denoted $t_i'$);
- Feng Luo (using a multiplicative structure on $SC(\Sigma)$) (denoted $t_i''$);
- Robert Penner (using the train tracks) (denoted $p_i$);
- Dylan Thurston (using the algebraic intersection number) (denoted $t_i$).

**Lemma**

*These definitions are related by:*

$$p_i = \frac{t_i + l(B, E; A) + l(D, E; C) - m_i}{2}$$  

$$t_i' = t_i'' = \frac{t_i}{2}.$$
Let $S$ be the thrice punctured sphere

$$S = \hat{\mathbb{C}} - \{ P^1, P^2, P^3 \}$$

and $\rho : \mathbb{H} \to S$ be the holomorphic universal covering map. By conjugation, we may take the covering group of $\rho$ to be $F = \langle A, B \rangle \subset \text{PSL}(2, \mathbb{Z})$ where

$$A = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}.$$  

WLOG, we can assume

$$\rho(\infty) = P^1, \quad \rho(0) = P^2, \quad \rho(1) = P^3.$$
The horocyclic coordinates

We define $f : N \rightarrow \mathbb{C}$ by $f(z) = e^{i \pi \rho^{-1}(z)}$ where we choose $\rho^{-1}$ so that $f$ maps $c(P^1, P^2) \cap N$ into the open unit interval. A maximal domain of definition is $D = S \cup P^1 - c(P^2, P^3)$.

$f$ defines an isometry: $(D - \{P^1\}, d) \rightarrow (\Delta, ds)$ where $ds$ is the hyperbolic metric on the punctured unit disc $\Delta$.

**Definition**

Any analytic continuation of $f$ is hence called **horocyclic coordinate at $P^1$ relative to $P^2$.**

**Corollary**

- If $f_1$ is the horocyclic coordinate at $P^1$ relative to $P^3$, then $f_1 = -f$.
- If $f_2$ is the horocyclic coordinate at $P^2$ relative to $P^1$, then $(\log f)(\log f_2) = -\pi^2$
The tame plumbing construction

Let $S_0 = S$ and for $0 < r < e^{-\frac{\pi}{2}}$, let

$$S_r = S - \{ P \in S : 0 < \|z(P)\| \leq r \}.$$ 

Given $S^1$ and $S^2$ with horocyclic coordinates $z$ and $w$. For $0 < \|t\| \leq e^{-\pi}$, we construct $S_t = \frac{S^1_r \sqcup S^2_r}{\sim}$ where

$$S^1_r \ni P \sim Q \in S^2_r \iff z(P)w(Q) = t.$$ 

**Figure:** The construction of the surface $S_t$
Developing map

Associated to every complex projective structure on a compact surface $\Sigma$, there is the pair $(D, \rho)$ where $D$ is a map $D : \tilde{\Sigma} \to \hat{\mathbb{C}}$ is the developing map and the homomorphism $\rho : \Pi_1(\Sigma) \to \text{PSL}(2, \mathbb{C})$, equivariant with respect to $D$, is the holonomy representation.

$D$ is defined up to composition with elements of $\text{PSL}(2, \mathbb{C})$, while $\rho$ up to conjugation by elements of $\text{PSL}(2, \mathbb{C})$. 

\[ S \Phi_i \Phi_j \Phi_i \Phi_i^{-1} \]
Fix \( M \) endowed of a complex projective structure, that is there is \( \{U_i\}_{i \in I} \), an open covering of \( M \), and \( \{\Phi_i : U_i \rightarrow \hat{\mathbb{C}}\}_{i \in I} \). We will denote \( V_i := \Phi_i(U_i) \).

For every charts \( \Phi_i : U_i \rightarrow V_i \) and \( \Phi_j : U_j \rightarrow V_j \) such that \( U_i \cap U_j \neq \emptyset \), you can define an overlap map

\[
R = \Phi_i \circ \Phi_j^{-1} = \Phi_j(U_i \cap U_j) \rightarrow \Phi_i(U_i \cap U_j)
\]

by \( R(z) := \Phi_i \circ \Phi_j^{-1}(z) \).

This map is the restriction of an element of \( \text{PSL}(2, \mathbb{C}) \).

Using that function, we could map the set \( V_j \in \hat{\mathbb{C}} \) into the set \( R(V_j) \) such that \( V_i \cap R(V_j) \neq \emptyset \).

Given a path \( \gamma \) in \( \Sigma \), we could cover it with \( U_0, \ldots, U_n \). So the holonomy of \( \gamma \in \pi_1(\Sigma) \) will be \( \rho(\gamma) = R_0 \cdots R_{n-1} \), where \( R_i \) are the overlap maps.
The standard chart $\Phi_i : P_i \rightarrow \mathbb{H}$

Fix a pants decomposition $\mathcal{P} = \{\gamma_1, \ldots, \gamma_\xi\}$ of the surface and cut $\Sigma$ along the pants curves into $k$ pants $P_1, \ldots, P_k$.

Choose, now, a "standard" pair of pants $P$.

For each pant $P_i$ with $i = 1, \ldots, k$ fix a standard chart $\Phi_i : P_i \rightarrow \mathbb{H}$ as follows:

- Label $\partial P_i$ as $0, 1, \infty$ in some order. That means to choose for each $P_i$ an orientation-preserving homeomorphism $f_i : P \rightarrow P_i$ where $P$ is the fixed "standard" pair of pants.
- Fix oriented geodesics joining the components of $\partial P_i$. 

\[
\begin{array}{c}
\epsilon_1 = 0 \\
\epsilon_2 = 1 \\
\epsilon_3 = \infty \\
\end{array}
\]
To glue \((P_i, \epsilon_i)\) and \((P_j, \epsilon_j)\) where \(\epsilon_j, \epsilon_j \in \{0, 1, \infty\}\) we have to:

- Label the incoming and the outgoing oriented edges \(\lambda_i\) and \(\lambda_j\).
- Map each pair of pants \(P_i\) and \(P_j\) to \(\mathbb{H}\) by standard charts \(\Phi_i : P_i \rightarrow \mathbb{H}\) and \(\Phi_j : P_j \rightarrow \mathbb{H}\).
- Map each side by standard maps \(\Omega_i : \mathbb{H} \rightarrow \mathbb{H}\) and \(\Omega_j : \mathbb{H} \rightarrow \mathbb{H}\) in such a way to take \(\epsilon_i\) and \(\epsilon_j\) to \(\infty\) and oriented lines to oriented lines.
- Identify the pictures (or rather annuli round cusps) using a translation fixing \(\infty\):
  1. first do \(J : \mathbb{H} \rightarrow \mathbb{H}\) defined by \(J(z) = -z\) to get the same orientations;
  2. then apply the translation \(T_\tau : \mathbb{H} \rightarrow \mathbb{H}\) defined by \(T_\tau(z) = z + \tau\) for a suitable \(\tau \in \mathbb{C}\).

So the overlap map is of the form \(\Omega_i^{-1} J^{-1} T_i^{-1} \Omega_j'\).
Glueing along horocycles

\[ \Omega_i(z) \quad \Omega_i(e_i) = \infty \]

\[ \Omega'(w) \quad \Omega'(e'_i) = \infty \]

\[ J \]

\[ T_\tau \]
Glueing along horocycles

\[ \Omega_i \circ J \circ T(U_i) \]

\[ \Omega_i(U'_i) \]

\[ \Omega_i^{-1}J^{-1}T^{-1}\Omega'_i(U'_i) \]

\[ U'_i \]

**Figure:** The identification along horocycles and its translation

The trace formula for a representation of the fundamental group in $\text{PSL}(2,\mathbb{C})$
Example: Maskit embedding of the once punctured torus

We want to glue the cusp \((P, \infty)\) and \((P, 0)\).

- Consider the standard charts \(\Phi : P \longrightarrow \mathbb{H}\) and \(\Phi' : P \longrightarrow \mathbb{H}\).
- Consider the maps \(\Omega : \mathbb{H} \longrightarrow \mathbb{H}\) and \(\Omega' : \mathbb{H} \longrightarrow \mathbb{H}\) where \(\Omega = \text{Id}\), while \(\Omega'\) is defined by \(\Omega'(w) = 1 - \frac{1}{w}\).
- Consider the map \(F : \mathbb{H} \longrightarrow \mathbb{H}\) defined by \(F(z) = -z\) and the translation \(T_\tau : \mathbb{H} \longrightarrow \mathbb{H}\) defined by \(T_\tau(z) = z + \tau\).
- Identify \(T_\tau \circ F \circ \text{Id}\) with \(\Omega'\) to get:

\[
T_\tau \circ F(z) = \Omega'(w) = 1 - \frac{1}{w}
\]

\[
-z + \tau = 1 - \frac{1}{w}
\]

\[
z = \frac{1}{w} + \tau - 1.
\]

Hence this is the **Maskit map** \(\begin{pmatrix} i\mu & i \\ i & 0 \end{pmatrix}\) with \(\mu = \tau - 1\).
The maps $\Omega_i$

The map $\Omega_{\epsilon_i}$ where $\epsilon_i = 0, 1, \infty$ have to send $\epsilon_i$ to $\infty$.

There are exactly 3 maps fixing the $0, 1, \infty$-triangle $\Delta_0$: $\Omega_0$, $\Omega_1$ and $\Omega_\infty$ defined by the following matrixes in $\text{SL}(2, \mathbb{C})$:

- $\Omega_0 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$
- $\Omega_1 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$
- $\Omega_\infty = \text{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Instead, the map $J$ defined by $J(z) = -z$ is represented in $\text{SL}(2, \mathbb{C})$ by $\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$.
The trace formula for a representation of the fundamental group of a surface in $\text{PSL}(2, \mathbb{C})$.
Paths between $b_0$ and $b_1$

The paths between $b_0$ and $b_1$ are:

- $\gamma_0$ that connects $b_0$ and $b_1$ crossing the geodesic between 1 and $\infty$;
- $\gamma_1$ that connects $b_0$ and $b_1$ crossing the geodesic between 0 and $\infty$;
- $\gamma_\infty$ that connects $b_0$ and $b_1$ crossing the geodesic between 0 and 1.

The holonomy of these different paths is given by:

- $\rho(\gamma_0) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$;
- $\rho(\gamma_1) = \text{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;
- $\rho(\gamma_\infty) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. 

Sara Maloni

The trace formula for a representation of the fundamental group
Using the holonomy of these path, we can calculate the holonomy of the boundary loops:

- \[ \rho(\partial_\infty P) = \rho(\gamma_0 \gamma_1^{-1}) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} ; \]
- \[ \rho(\partial_0 P) = \rho(\gamma_1 \gamma_\infty^{-1}) = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} ; \]
- \[ \rho(\partial_1 P) = \rho(\gamma_\infty \gamma_0^{-1}) = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} . \]
### Types of crossing

<table>
<thead>
<tr>
<th>Name of crossing-type</th>
<th>Type of crossing</th>
<th>Matrix of crossing</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0 \rightarrow 0$</td>
<td>$\Omega^{-1}<em>0 J^{-1} T^{-1}</em>\tau \Omega_0$</td>
</tr>
<tr>
<td>2</td>
<td>$0 \rightarrow 1$</td>
<td>$\Omega^{-1}<em>0 J^{-1} T^{-1}</em>\tau \Omega_1$</td>
</tr>
<tr>
<td>3</td>
<td>$0 \rightarrow \infty$</td>
<td>$\Omega^{-1}<em>0 J^{-1} T^{-1}</em>\tau \Omega_\infty$</td>
</tr>
<tr>
<td>4</td>
<td>$1 \rightarrow 0$</td>
<td>$\Omega^{-1}<em>1 J^{-1} T^{-1}</em>\tau \Omega_0$</td>
</tr>
<tr>
<td>5</td>
<td>$1 \rightarrow 1$</td>
<td>$\Omega^{-1}<em>1 J^{-1} T^{-1}</em>\tau \Omega_1$</td>
</tr>
<tr>
<td>6</td>
<td>$1 \rightarrow \infty$</td>
<td>$\Omega^{-1}<em>1 J^{-1} T^{-1}</em>\tau \Omega_\infty$</td>
</tr>
<tr>
<td>7</td>
<td>$\infty \rightarrow 0$</td>
<td>$\Omega^{-1}<em>\infty J^{-1} T^{-1}</em>\tau \Omega_0$</td>
</tr>
<tr>
<td>8</td>
<td>$\infty \rightarrow 1$</td>
<td>$\Omega^{-1}<em>\infty J^{-1} T^{-1}</em>\tau \Omega_1$</td>
</tr>
<tr>
<td>9</td>
<td>$\infty \rightarrow \infty$</td>
<td>$\Omega^{-1}<em>\infty J^{-1} T^{-1}</em>\tau \Omega_\infty$</td>
</tr>
</tbody>
</table>
Using again the holonomy of the path between $b_0$ and $b_1$, we can calculate the holonomy of the archetypes:

- $\rho(L_\infty) = \rho(\gamma_\infty \gamma_1^{-1}) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$;
- $\rho(L_0) = \rho(\gamma_0 \gamma_\infty^{-1}) = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$;
- $\rho(L_1) = \rho(\gamma_1 \gamma_0^{-1}) = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$. 

Sara Maloni

The trace formula for a representation of the fundamental group of a surface in $PSL(2,\mathbb{C})$.
List of some possible blocks

<table>
<thead>
<tr>
<th>Block</th>
<th>Matrices in the block</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_{1-1}$</td>
<td>$D_0^\alpha D_1 D_0^{-\beta} \Gamma_1$ or $D_0^\alpha D_1^{-1} D_0^{-\beta} \Gamma_1$</td>
</tr>
<tr>
<td>$B_{1-6}$</td>
<td>$D_0^\alpha D_1^{-\beta} \Gamma_6$</td>
</tr>
<tr>
<td>$B_{3-7}$</td>
<td>$D_\infty^\alpha D_0 D_\infty^{-\beta} \Gamma_7$ or $D_\infty^\alpha D_0^{-1} D_\infty^{-\beta} \Gamma_7$</td>
</tr>
<tr>
<td>$B_{6-1}$</td>
<td>$D_\infty^\alpha D_0^{-\beta} \Gamma_1$</td>
</tr>
<tr>
<td>$B_{6-6}$</td>
<td>$D_\infty^\alpha D_1^{-\alpha} \Gamma_6$</td>
</tr>
<tr>
<td>$B_{6-8}$</td>
<td>$D_\infty^\alpha D_0 D_\infty^{-\beta} \Gamma_8$ or $D_\infty^\alpha D_0^{-1} D_\infty^{-\beta} \Gamma_8$</td>
</tr>
<tr>
<td>$B_{7-3}$</td>
<td>$D_0^\alpha D_1 D_0^{-\beta} \Gamma_3$ or $D_0^\alpha D_1^{-1} D_0^{-\beta} \Gamma_3$</td>
</tr>
<tr>
<td>$B_{7-7}$</td>
<td>$D_0^\alpha D_\infty^{-\beta} \Gamma_7$</td>
</tr>
<tr>
<td>$B_{8-6}$</td>
<td>$D_1^\alpha D_\infty D_1^{-\beta} \Gamma_6$ or $D_1^\alpha D_\infty^{-1} D_1^{-\beta} \Gamma_6$</td>
</tr>
<tr>
<td>$B_{8-8}$</td>
<td>$D_1^\alpha D_\infty^{-\beta} \Gamma_8$</td>
</tr>
</tbody>
</table>
The loop $\beta$

Figure: The curve $\beta$ in the four holed sphere
The loop $\beta$

We could split $\beta$ in four paths $\beta_1, \beta_2, \beta_3, \beta_4$.

- $\beta_1$ is the crossing from the “white” triangle to the “white” triangle with holonomy
  $$\rho(\beta_1) = \Omega_\infty^{-1} J^{-1} T_\tau^{-1} \Omega_\infty = J^{-1} T_\tau^{-1} = \begin{pmatrix} i & -i\tau \\ 0 & -i \end{pmatrix}.$$ 

- $\beta_2$ is the crossing of $c(0, 1)$ in $P'$ with holonomy
  $$\rho(\beta_2) = \rho(\gamma_\infty) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$ 

- $\beta_3$ is the crossing from the “black” triangle to the “black” triangle with holonomy $\rho(\beta_3) = (\Omega'_\infty)^{-1} J^{-1} T_{\tau-2}^{-1} \Omega'_\infty = J^{-1} T_{\tau-2}^{-1} = \begin{pmatrix} i & -i(\tau - 2) \\ 0 & -i \end{pmatrix}.$$

- $\beta_4$ is the crossing of $c(0, 1)$ in $P$ with holonomy
  $$\rho(\beta_4) = [\rho(\gamma_\infty)]^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$
The loop $\beta$

Hence

$$\rho(\beta) = \rho(\beta_1) \circ \rho(\beta_2) \circ \rho(\beta_3) \circ \rho(\beta_4)$$

$$= \begin{pmatrix} i & -i\tau \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} i & -i(\tau - 2) \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$= -\begin{pmatrix} -4\tau^2 - 10\tau + 5 & 2\tau^2 - 4\tau + 3 \\ -4\tau - 12 & 2\tau - 3 \end{pmatrix}$$

So $\text{Tr}(\rho(\beta)) = 4\tau^2 + 8\tau - 2$. 

Sara Maloni

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Example: the four holed sphere

Trace formula
Theorem (Trace formula)

Let $\gamma$ be a simple closed curve on the surface $\Sigma$ and let $i(\gamma) = (m_1, t_1, \cdots, m_\xi, t_\xi)$ be the Dehn-Thurston coordinates of $\gamma$ w.r.t. $\mathcal{P}$ where $\sum_{i=1}^\xi m_i \neq 0$. Let $\rho : \Pi_1(\Sigma) \to \text{PSL}(2, \mathbb{C})$ be the holonomy representation corresponding to glueing around horocycles with parameters $\tau_1, \cdots, \tau_\xi$ and let $\gamma$ be represented by $g \in \Pi_1(\Sigma)$. Then $\text{Tr} \rho(g)$ is a polynomial in $\tau_1, \cdots, \tau_\xi$ such that:

$$\text{Tr} \rho(g) = \pm i m 2^h (\tau_1 - \frac{(t_1 - m_1)}{m_1})^{m_1} \cdots (\tau_\xi - \frac{(t_\xi - m_\xi)}{m_\xi})^{m_\xi} + O(m - 2)$$

where $m = \sum_{i=1}^\xi m_i$, $h = \#\{\text{archetypes in } \gamma\}$ and where $O(m - 2)$ means terms of degree at most $m_i$ in the variable $\tau_i$, but with total degree in $\tau_1 \cdots \tau_\xi$ at most $m - 2$.  

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The trace formula for a representation of the fundamental group.
We have a lot of ideas for further developing:

- Use this trace formula to study the asymptotic direction of the pleating variety $P_\eta$, where $\eta \in M\mathcal{L}(\Sigma)$, in the image $M_g^b$ of the Maskit embedding $m : \text{Teich}(\Sigma_g^b) \to \mathbb{C}^\xi$.

- Generalize the results obtain by Raquel Águeda Maté in her Ph.D. Thesis for the Schottky Space of genus 2 to study the Schottky Space of genus $g$.

- ... Suggestions?!? ...