Dehn-Thurston coordinates.

Given a surface $\Sigma$, a pants decomposition of $\Sigma$ is a maximal set $\mathcal{P} = \{\sigma_1, \ldots, \sigma_r\}$ of homotopically distinct and non-boundary parallel loops.

Fix a pants decomposition $\mathcal{P}$ and a marking on it, that is dual curves $D_i$ such that $i(D_i, \sigma_i) = 2\delta_i$. Then, given a curve $\alpha$, for each pants curve $\sigma_i$ we define:

- the length parameter $p_i(\alpha) = i(\alpha, \sigma_i)$;
- the twist parameter $q_i(\alpha) = i(\alpha \cap A_i, D_i \cap A_i)$.

where $i$ is the geometric, while $\epsilon$ is the algebraic intersection number.

### Projective structure and plumbing construction

A complex projective structure on a compact surface $\Sigma$ is the pair $(\Delta, \rho)$ where $\Delta$ is a map $D : \Sigma \to \tilde{C}$ from the universal covering space $\Sigma$ to $\tilde{C}$ called the developing map and $\rho$ is a homomorphism $\rho : \pi_1(\Sigma) \to \text{PSL}(2, \mathbb{C})$ equivariant with respect to $D$ called the holonomy representation.

Kra's plumbing construction endows $\Sigma$ with a projective structure as follows. Glue, or 'plumb', adjacent pants as described in the recipe below. The gluing across the $i$th pants curve is defined by $\tau_i \in \mathbb{C}$. The associated holonomy representation $\rho : \pi_1(\Sigma) \to \text{PSL}(2, \mathbb{C})$ gives a projective structure on $\Sigma$ which depends holomorphically on the $\tau_i$.

### Gluing cusps around horocycles.

To glue $(P_i, \epsilon_i)$ and $(P_j, \epsilon_j)$ where $\epsilon_i, \epsilon_j \in \{0, 1, \infty\}$ we have to:

1. Label the incoming and the outgoing oriented edges $\lambda_i$ and $\lambda_j$.
2. Map each pair of pants $P_i$ and $P_j$ to $\mathbb{H}$ by standard charts $\Phi_i : P_i \to \mathbb{H}$ and $\Phi_j : P_j \to \mathbb{H}$.
3. Map each side by standard maps $\Omega_i : \mathbb{H} \to \mathbb{H}$ and $\Omega_j : \mathbb{H} \to \mathbb{H}$ in such a way to take $\epsilon_i$ and $\epsilon_j$ to $\infty$ and oriented lines to oriented lines.
4. Identify the pictures (or rather annuli round cusps) using a translation fixing $\infty$:
   1. first do $j : \mathbb{H} \to \mathbb{H}$ defined by $f(z) = -\bar{z}$ to get the same orientations;
   2. then apply the translation $T_\tau : \mathbb{H} \to \mathbb{H}$ defined by $T_\tau(z) = z + \tau_i$ for a suitable $\tau_i \in \mathbb{C}$.

So the overlap map is of the form:

$$\Omega_j^{-1} T_\tau^{-1} \Omega_i.$$

### The Maskit embedding

- **Figure**: The Maskit embedding of the once punctured torus

The Maskit embedding is a polynomial in the Maskit embedding.

### Main theorem: the trace formula

The main result of this paper is a very simple relationship between the coefficients of the top terms of $\rho(\gamma)$, as polynomials in the $\tau_i$, and the Dehn-Thurston coordinates of $\gamma$ relative to the pants decomposition $\mathcal{P}$.

Let $\gamma$ be a connected simple closed curve on the hyperbolic surface $\Sigma$, not parallel to any of the pants curves $\sigma_i$. Let $\rho$ be the holonomy representation defined by Kra's plumbing construction. Then $\text{Tr} \rho(\gamma)$ is a polynomial in $\tau_1, \ldots, \tau_r$ whose top terms are given by:

$$\text{Tr} \rho(\gamma) = \pm \frac{p_2(q_1, \ldots, q_r)}{q_1} \cdots \frac{p_l(q_1, \ldots, q_r)}{q_l} + R,$$

where

- $q = \sum q_i > 0$;
- $R$ represents terms with total degree in $\tau_1, \ldots, \tau_r$ at most $q - 2$ and of degree at most $q_i$ in the variable $\tau_i$;
- $h = h(\gamma)$ is the total number of sec-arcs in the standard representation of $\gamma$ relative to $\mathcal{P}$, see below.

If $q = 0$, then $\gamma = \sigma_i$ for some $i$, $\rho(\gamma)$ is parabolic, and $\text{Tr} \rho(\gamma) = \pm 2$.

The trace formulae could be used to find the asymptotic directions of pleating rays in the Maskit embedding.

### Maskit embedding

If the representation $\rho$ described above is free and discrete, then the resulting hyperbolic 3-manifold $M = \mathbb{H}^3/\rho(\pi_1(\Sigma))$ lies on the boundary of the quasifuchsian space $\Omega \mathcal{T}(\Sigma)$. One end of $M$ consists of a union of triply punctured spheres obtained by pinching in $\Sigma$ the curves $\sigma_i$, defining $\mathcal{P}$. Suppose that, in addition, $\rho(\pi_1(\Sigma))$ is geometrically finite and that the other end $\Omega \mathcal{T}(\rho(\pi_1(\Sigma)))$ of $M$ is a Riemann surface homeomorphic to $\Sigma$. Since the triply punctured spheres are rigid, it follows that the Riemann surface structure of $\Omega \mathcal{T}(\rho(\pi_1(\Sigma)))$ runs over the Teichmüller space $\mathcal{T}(\Sigma)$ of $\Sigma$. The image of the space of all such groups in the character variety $\mathcal{R}$ of $\Sigma$ is called the Maskit embedding of $\mathcal{T}(\Sigma)$, see figure below.

Our construction, for suitable values of the parameters $\tau_i$, gives exactly the Maskit embedding of $\mathcal{T}(\Sigma)$.

### References