Plumbing Constructions in Quasifuchsian Space.

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Dehn–Thurston coordinates

Given a pants decomposition $\mathcal{P} = \{\sigma_1, \ldots, \sigma_\xi\}$ on a surface $\Sigma$, Dehn defined an injection $i : S = S(\Sigma) \longrightarrow \mathbb{Z}_{\geq 0}^\xi \times \mathbb{Z}^\xi$ by

$$i(\gamma) = (q_1(\gamma), \ldots, q_\xi(\gamma); tw_1(\gamma), \ldots, tw_\xi(\gamma)).$$

1. $q_i(\gamma) = i(\gamma, \sigma_i) \in \mathbb{Z}_{\geq 0}$ are the length parameters;
2. $tw_i(\gamma) \in \mathbb{Z}$ are the twist parameters of $\gamma$.

**Figure:** Penner and Harer twist $\hat{p}_i = -1$ and D. Thurston’s twist $p_i = 0$. 

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**DT coordinates**

**Maskit embedding**

**Gluing construction**

**Main theorems**

**Other slices**
Relation between \( \hat{p}_i \) and \( p_i \)

Suppose two pairs of pants meet along \( \sigma = E \in \mathcal{PC} \). Label their respective boundary curves \((A, B, E)\) and \((C, D, E)\) in clockwise order.

**Theorem (M–Series)**

Let \( \gamma \in S \) and let \( \hat{p}_i \) and \( p_i \) denote the PH–twist and the DT–twist around \( \sigma \). Then

\[
\hat{p}_i = \frac{p_i + l(A, E; B) + l(C, E; D) - q_i}{2},
\]

where \( l(X, Y; Z) \) denotes the number of strands of \( \gamma \cap P \) running from the boundary curve \( X \) to the boundary curve \( Y \) in the pair of pants \( P = (X, Y, Z) \).
Let $\tau_{\text{Th}}$ be Thurston symplectic form on $S \subset \text{ML}_\mathbb{Q}(\Sigma)$.

**Theorem (M.)**

Suppose that loops $\gamma, \gamma' \in S$ belongs to the same chart and let $i(\gamma) = (q_1, \ldots, q_\xi; p_1, \ldots, p_\xi)$, $i(\gamma') = (q'_1, \ldots, q'_\xi; p'_1, \ldots, p'_\xi)$ their DT coordinates. Then

$$\tau_{\text{Th}}(\gamma, \gamma') = \frac{1}{2} \sum_{i=1}^{\xi} (q_ip'_i - q'_ip_i).$$

In addition, if $\gamma, \gamma'$ are disjoint, then $\tau_{\text{Th}}(\gamma, \gamma') = 0$. 
Basic definitions on Kleinian groups

PSL(2, \mathbb{C}) acts on \mathbb{H}^3 by isometries and on \hat{\mathbb{C}} = \mathbb{C} \cup \infty by conformal maps.

**Definition**

- A **Kleinian group** \( G \) is a discrete (torsion-free) subgroup of PSL(2, \mathbb{C}).
- The **limit set** \( \Lambda(G) \) is the set of accumulation points of the action of \( G \) on \( \hat{\mathbb{C}} \).
- The **regular set** \( \Omega(G) \) is \( \hat{\mathbb{C}} - \Lambda(G) \).
- A **Fuchsian group** is a discrete subgroup of PSL(2, \mathbb{R}), or, equivalently, a Kleinian group \( G \) such that \( \Lambda(G) \) is a circle.
- A **Quasifuchsian group** is a Kleinian group \( G \) such that \( \Lambda(G) \) is a topological circle, or, equivalently, a quasi-conformal deformation of a Fuchsian group.
The Maskit embedding

The **Maskit slice** $\mathcal{M}$ is the set of representations $\rho : \pi_1(\Sigma) \to PSL(2, \mathbb{C})$ (up to conjugation in $PSL(2, \mathbb{C})$) such that:

1. $G_\rho = \rho(\pi_1(\Sigma))$ is discrete and $\rho$ is an isomorphism,
2. $\rho(\sigma_i)$ are parabolic,
3. all components of $\Omega(G)$ are simply connected and there is exactly one invariant component $\Omega^+(G)$,
4. $\Omega^+(G)/G$ is homeomorphic to $\Sigma$ and the other components are triply punctured spheres.

**Figure: Quasifuchsian Group and Maskit Group.**
Picture of the Maskit embedding for the once punctured torus $\Sigma_{1,1}$

**Figure:** The Maskit embedding $\mathcal{M}(\Sigma_{1,1})$ for the once punctured torus. Picture courtesy David Wright.
A **pleated surface** is a hyperbolic surface which is totally geodesic almost everywhere and such that the locus of points where it fails to be totally geodesic is a geodesic lamination.

By Thurston, each component of the boundary $\partial C(G)/G$ of the convex core is a pleated surface.

Given $\rho \in \mathcal{M}$, denote $\beta(\rho) \in \text{ML}(\Sigma)$ the bending lamination of $\partial C^+/G$, where $G_\rho = \rho(\pi_1(\Sigma))$.

Given $[\eta] \in \text{PML}(\Sigma)$, the **pleating ray** $\mathcal{P} = \mathcal{P}[\eta]$ of $[\eta]$ is the set of elements $\rho \in \mathcal{M}$ for which $\beta(\rho) \in [\eta]$. 
Let $\Sigma$ be a surface with $\chi(\Sigma) < 0$ and let $\mathcal{PC} = \{\sigma_1, \ldots, \sigma_\xi\}$ be a pants decomposition on it. Let $\mu = (\mu_1, \ldots, \mu_\xi) \in \mathbb{H}^\xi$.

**STEP 1:** Any triply punctured sphere is isometric to $\mathbb{P} = \mathbb{H}/\Gamma$, where

$$\Gamma = \langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \rangle.$$

Identify any $P_i$ to the fundamental domain $\Delta$ of $\Gamma$ by the homeomorphisms

$$\Phi_i : \text{int}(P_i) \longrightarrow \Delta.$$
**Gluing construction**

*STEP 2:* Let $\sigma_i = \partial_\epsilon P \cap \partial_\epsilon' P'$, then the gluing is described by

$$
\Omega_\epsilon^{-1} J^{-1} T_{\mu_i}^{-1} \Omega_{\epsilon'}
$$

where $\mu_i \in \mathbb{H}$ is the **gluing parameter** and $\Omega_\infty = \text{Id}$,

$$
\Omega_0 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Omega_1 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},
$$

$$
J = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad T_{\mu_i} = \begin{pmatrix} 1 & \mu_i \\ 0 & 1 \end{pmatrix}.
$$
This describes a (complex) projective structure on $\Sigma$, which depends on the gluing parameter $\mu = (\mu_1, \ldots, \mu_\xi) \in \mathbb{H}^\xi$. In particular, given $\mu \in \mathbb{H}^\xi$, we define a developing map $\text{Dev}_\mu : \tilde{\Sigma} \longrightarrow \hat{\mathbb{C}}$ and a holonomy map $\rho_\mu : \pi_1(\Sigma) \longrightarrow PSL(2, \mathbb{C})$.

**Theorem (M–Series)**

If $\text{Dev}_\mu : \tilde{\Sigma} \longrightarrow \hat{\mathbb{C}}$ is an embedding, then $\rho_\mu$ is a group isomorphism and $\rho_\mu \in \mathcal{M}$.

In addition, these representations $\rho_\mu$ parametrise $\mathcal{M}$.
Let $\rho_\mu: \pi_1(\Sigma) \rightarrow PSL(2, \mathbb{C})$ be the holonomy described by the gluing construction. Let $\gamma$ be a simple closed curve on $\Sigma$, not parallel to any of the pants curves $\sigma_i$.

**Theorem (Top Terms’ Formula, M – Series)**

$$
\text{Tr} \rho_\mu(\gamma) = \pm i^q 2^h \left( \mu_1 + \frac{(p_1 - q_1)}{q_1} \right)^{q_1} \cdots \left( \mu_\xi + \frac{(p_\xi - q_\xi)}{q_\xi} \right)^{q_\xi} + R,
$$

where

- $q = \sum_{i=1}^\xi q_i > 0$;
- $R$ represents terms with total degree in $\mu_1 \cdots \mu_\xi$ at most $q - 2$;
- $h = h(\gamma)$ is the total number of waves.
Asymptotic direction of pleating rays

Theorem (Asymptotic direction, M, Series, Keen–Series)

Suppose that $\eta = \sum_{i=1}^{\xi} a_i \gamma_i$ is an admissible measured lamination on $\Sigma$. Then, as the bending measure $\beta(G_{\mu}) \in \eta$ tends to zero, the pleating ray $P[\eta]$ in $\mathcal{M}$ approaches the line

$$\Re \mu_i = -\frac{p_i(\eta)}{q_i(\eta)} + 1, \quad \Im \mu_1 = \frac{q_j(\eta)}{q_1(\eta)},$$

where $(q_1(\eta), \ldots, q_\xi(\eta); p_1(\eta), \ldots, p_\xi(\eta))$ are the Dehn–Thurston coordinates for $\eta$. 
Generalised gluing construction

Given a pants decomposition $\mathcal{PC} = \{\sigma_1, \ldots, \sigma_\xi\}$ on $\Sigma$, let $c = (c_1, \ldots, c_\xi) \in \mathbb{R}_+^\xi$ and $\mu = (\mu_1, \ldots, \mu_\xi) \in (\mathbb{C}/2i\pi)^\xi$. We describe a (complex) projective structure on $\Sigma$ with developing map $\text{Dev}_{c,\mu}$ and holonomy map $\rho_{c,\mu}$. In particular, $\rho_{c,\mu}(\gamma)$ is hyperbolic and $\text{Tr} \rho_{c,\mu}(\gamma) = \pm 2 \cosh(c_j)$.

**Theorem (M.)**

If $c \rightarrow 0$ keeping $\mu$ fixed, where $\mu_i = \frac{i\pi - \mu_i}{c_i}$, then

$$\rho_{c,\mu} \rightarrow \rho_{\mu}.$$
Linear slices $\mathcal{L}_c$

Given $\mathcal{PC} = \{\sigma_1, \ldots, \sigma_\xi\}$, the complex Fenchel–Nielsen coordinates $\text{FN}_C : \mathcal{QF}(\Sigma) \longrightarrow (\mathbb{C}_+/2i\pi)^\xi \times (\mathbb{C}/2i\pi)^\xi$ are defined by

$$\text{FN}_C(G) = (\lambda_1, \ldots, \lambda_\xi, \tau_1, \ldots, \tau_\xi),$$

where $\lambda_i$ are the complex length and $\tau_i$ are the complex twist of the pants curve $\sigma_i$.

**Definition**

Given $c \in \mathbb{R}_+^\xi$, we define the $c$–slice (or the linear slice) $\mathcal{L}_c$ to be the set

$$\mathcal{L}_c = \{(c, \tau) \in \text{FN}_C (\mathcal{QF}(\Sigma)) \mid \text{sign}(\Im \tau_1) = \ldots = \text{sign}(\Im \tau_\xi)\}.$$
Connectedness of linear slices $\mathcal{L}_c$

Figure: The linear slice $\mathcal{L}_c$ when $c = 1, 2, 4, 5, 10, 20.$
Connectedness of linear slices $\mathcal{L}_c$