Polyhedra inscribed in a hyperboloid and anti-de Sitter geometry.

Jeffrey Danciger $^1$  Sara Maloni $^2$  Jean-Marc Schlenker $^3$

$^1$University of Texas at Austin
$^2$Brown University
$^3$University of Luxembourg

AMS Sectional Meeting, UMBC
March 28, 2014
Historical introduction

Question (Steiner (1832))

What are the graphs obtained as 1-skeletons of a convex polyhedron in $\mathbb{R}^3$?

Theorem (Steinitz (1916))

An embedded graph in the sphere $S^2$ is the 1-skeleton of a convex polyhedron in $\mathbb{R}^3$ if and only if it is 3-connected (that is, suppressing 2 vertices leaves a connected graph).

Question (Steiner (1832))

Which ones are obtained from polyhedra inscribed in $S^2$?
Polyhedra inscribed in a sphere

Theorem (Steinitz (1927))

Some of those combinatorics cannot be realized by polyhedra inscribed in a sphere.

Theorem (Hodgson-Rivin-Smith (1992))

The answer depends on the existence of a solution to a set of linear equations and inequalities. (It can be decided in polynomial time.)

Question

What about polyhedra inscribed in a hyperboloid?
Our results

Let $\Gamma$ be a graph embedded in $S^2$, we call $S_\Gamma$ [resp. $H_\Gamma$] the space of convex polyhedra inscribed in the sphere [resp. in the hyperboloid] with 1-skeleton $\Gamma$, up to projective transformations leaving the sphere [resp. hyperboloid] invariant.

Theorem (Danciger-M.- Schlenker)

$H_\Gamma \neq \emptyset \iff S_\Gamma \neq \emptyset$ and $\Gamma$ admits a Hamiltonian cycle.

Theorem (Danciger-M.- Schlenker)

(i) If $H_\Gamma \neq \emptyset$, then $S_\Gamma \neq \emptyset$.
(ii) If $S_\Gamma \neq \emptyset$, then \{c.c. of $H_\Gamma$\} $\longleftrightarrow$ \{Hamiltonian cycles in $\Gamma$\}.
Let $\Gamma$ be a graph embedded in $S^2$, and let $\Gamma^*$ be the graph dual to $\Gamma$.

**Theorem (Rivin (1992))**

Let $\theta : \Gamma^* \to (0, \pi)$. There is a non-planar convex ideal polyhedron in $\mathbb{H}^3$ with combinatorics given by $\Gamma$ and exterior dihedral angles given by $\theta$ if and only if:

(i) For any simple closed curve $c$ in $\Gamma^*$ bounding a face, the sum of the values of $\theta$ on the edges of $c$ is $2\pi$.

(ii) For any simple closed curve $c$ in $\Gamma^*$ not bounding a face, the sum of the values of $\theta$ on the edges of $c$ is $> 2\pi$.

Rivin extended a result proved by Andreev (1970) for compact and ideal polyhedra $P$ of finite volume with dihedral angles $\leq \pi/2$. 
Ideal hyperbolic polyhedra

Induced metrics

**Theorem (Rivin (1992))**

*Any complete hyperbolic metric of finite area on the sphere minus N points, with N ≥ 3, is induced on a unique ideal hyperbolic polyhedron.*

Rivin extended a result proved by Alexandrov (1944-50) for compact polyhedra.
Anti-de Sitter geometry

The Anti-de Sitter space $\text{AdS}^3$ is

$$\text{AdS}^3 = \{ x \in \mathbb{R}^{2,2} : \langle x, x \rangle_{2,2} < 0 \} / \sim ,$$

where $x \sim y$ $\iff$ $\exists \lambda \in \mathbb{R}_+ \text{ such that } x = \lambda y$, with the induced Lorentzian metric. Its isometry group is $\text{PO}(2, 2)$. The ideal boundary is

$$\partial_\infty \text{AdS}^3 = \{ x \in \mathbb{R}^{2,2} : \langle x, x \rangle_{2,2} = 0 \} / \sim .$$

A convex ideal AdS polyhedron is a convex polyhedron in $\text{AdS}^3$ with its vertices on the ideal boundary. The faces of an ideal polyhedron are space-like. The dihedral angles along the edges of the equator (called exterior) are in $(-\infty, 0)$, while the other dihedral angles lie in $(0, +\infty)$. 
Dihedral angles

Let $\Gamma^*$ be the graph dual to $\Gamma$.

**Theorem (Danciger-M.-Schlenker)**

Let $\theta : \Gamma^* \rightarrow \mathbb{R}_{\neq 0}$. There is a non-planar convex ideal AdS polyhedron with combinatorics given by $\Gamma$ and exterior dihedral angles given by $\theta$ if and only if:

(i) The edges of $\Gamma$ on which $\theta < 0$ form a Hamiltonian cycle in $\Gamma$.

(ii) For any simple closed curve $c$ in $\Gamma^*$ bounding a face, the sum of the values of $\theta$ on the edges of $c$ is zero.

(iii) For any simple closed curve $c$ in $\Gamma^*$ not bounding a face, and containing at most two edges where $\theta < 0$, the sum of the values of $\theta$ on the edges of $c$ is positive.
Theorem (Danciger-M.- Schlenker)

Let \( m \) be a finite-volume hyperbolic metric on \( S^2 \) with \( N \) cusps, and let \( e \) be a closed path going through each vertex exactly once. Then there is a unique ideal polyhedron \( P \subset \text{AdS}^3 \) (up to global isometry) so that the induced metric on \( P \) is isometric to \( m \) and its path of external edges is homotopic to \( e \).
Now we will prove the following:

**Theorem (Danciger-M.- Schlenker)**

\[ \mathcal{H}_\Gamma \neq \emptyset \iff S_\Gamma \neq \emptyset \text{ and } \Gamma \text{ admits a Hamiltonian cycle.} \]

Let \( P \in S_\Gamma \) and let \( \gamma \) be an Hamiltonian cycle for \( \Gamma \). Let \( \theta : \Gamma_1 \rightarrow (0, \pi) \) be its dihedral angles which satisfies the conditions of Rivin’s theorem on dihedral angles. Then we can define \( \theta' : \Gamma_1 \rightarrow \mathbb{R} \neq 0 \) by

- \( \theta'(e) = \theta(e) \) if is not an edge of \( \gamma \),
- \( \theta'(e) = \theta(e) - \pi \) if is an edge of \( \gamma \).

Then \( \theta' : \Gamma_1 \rightarrow \mathbb{R} \) satisfies the conditions of our theorem on dihedral angles. Therefore \( \mathcal{H}_\Gamma \neq \emptyset \).
Let $P \in \mathcal{H}_\Gamma$. Let $\theta : \Gamma_1 \longrightarrow \mathbb{R} \neq 0$ be its dihedral angles, and let $\gamma$ be the cycle of its exterior edges. We can choose $t > 0$ such that:

- $\forall e \in \Gamma_1$ of $\Gamma$, $t\theta(e) \in (-\pi, \pi)$;
- $\forall$ s. c. c. $c$ in $\Gamma^*$ not bounding a face, and intersecting $\gamma$ in $k$ points, then the sum of the values of $t\theta$ on the edges of $c$ is $> (2 - k)\pi$.

Moreover, $t\theta(e) < 0 \iff e$ is an edge of $\gamma$.

Let $\theta' : \Gamma_1 \longrightarrow (0, \pi)$ be defined by:

- $\theta'(e) = t\theta(e)$ if is not an edge of $\gamma$,
- $\theta'(e) = \pi + t\theta(e)$ if is an edge of $\gamma$.

Then $\theta' : \Gamma_1 \longrightarrow (0, \pi)$ satisfies the conditions of Rivin’s theorem on dihedral angles. Therefore $S_\Gamma \neq \emptyset$. 

Danciger, Maloni, Schlenker
Proof of the theorem on dihedral angles

Definitions

Let \( \Gamma \) be a 3-connected graph embedded in \( S^2 \), and let \( \gamma \) be an simple closed curve in \( \Gamma \) going through each vertex.

- We call \( \mathcal{A}_{\Gamma,\gamma} \) the space of maps \( \theta : \Gamma_1 \to \mathbb{R} \) such that:
  - for all \( e \in \Gamma \), \( \theta(e) < 0 \) if \( e \) is in \( \gamma \), \( \theta(e) > 0 \) otherwise,
  - the sum of the values of \( \theta \) on the boundary of any face of \( \Gamma^* \) is zero,
  - the sum of the values of \( \theta \) on any other cycle in \( \Gamma^* \) intersecting \( \gamma \) at most twice is positive.

- We denote by \( \mathcal{P}_{\Gamma,\gamma} \) the space of polyhedral embeddings of \( S^2 \) in \( \text{AdS}^3 \) with image an ideal polyhedron with 1-skeleton \( \Gamma \) such that the cycle of exterior edges is \( \gamma \).

- The map \( \Psi_{\Gamma,\gamma} : \mathcal{P}_{\Gamma,\gamma} \to \mathcal{A}_{\Gamma,\gamma} \) sends a polyhedron to its exterior dihedral angles.
Proof of the theorem on dihedral angles

**Sketch of the proof**

**Lemma (Danciger-M.- Schlenker)**

\[ \Psi_{\Gamma, \gamma} \text{ is a proper local homeomorphism.} \]

(Hence \( \Psi_{\Gamma, \gamma} : \mathcal{P}_{\Gamma, \gamma} \rightarrow \mathcal{A}_{\Gamma, \gamma} \) is a covering.)

**Lemma (Danciger-M.- Schlenker)**

1. As \( t \rightarrow 0 \), \( P_t \) converges to a flat polyhedron \( P_0 \).
2. For any \( \Gamma \) and \( \gamma \) such that \( \mathcal{A}_{\gamma, \Gamma} \neq \emptyset \), \( \exists \) a nbhd \( U \) of \( \mathcal{P}_{\gamma, \gamma}^0 \) and a nbhd \( V \) of 0 in \( \mathcal{A}_{\gamma, \Gamma} \) s. t. \( \Psi_{\gamma, \Gamma}|_U : U \rightarrow V \) is a homeomorphism.

- \( \exists \) a nbhd \( U \) of \( \mathcal{P}_{\gamma, \gamma}^0 \) in \( \mathcal{P}_{\gamma, \Gamma} \) and a nbhd \( V \) of 0 in \( \mathcal{A}_{\gamma, \Gamma} \) s. t. \( \forall \theta \in V \) has a unique inverse image in \( U \) by \( \Psi_{\gamma, \Gamma} \).
- Any \( \theta \in V \) can have inverse images only in \( U \).

(Hence \( \Psi_{\Gamma, \gamma} : \mathcal{P}_{\Gamma, \gamma} \rightarrow \mathcal{A}_{\Gamma, \gamma} \) has degree one, so is a homeo.)
End
Proof of the theorem on induced metrics

Definitions

- By $\mathcal{P}_\gamma$ the space of ideal polyhedral embeddings of $S^2$ in $\text{AdS}^3$ with vertices exactly at the $v_i$, with $\gamma$ isotopic to the (oriented) equator, considered up to global isometry.

- By $\mathcal{M}_\gamma$ the space of complete hyperbolic metrics on the sphere $S^2$ with cusps at the $v_i$, marked by the closed curve $\gamma$, considered up to isotopy.

- By $\Phi_\gamma : \mathcal{P}_\gamma \rightarrow \mathcal{M}_\gamma$ the map sending an ideal $\text{AdS}$ polyhedral embedding to its induced metric.
Sketch of the proof

**Lemma (Danciger-M.- Schlenker)**

ϕ_γ is a proper local homeomorphism.

(Hence ϕ: P → M is a covering.)

**Lemma (Danciger-M.- Schlenker)**

P_γ is connected, and M_γ is connected and simply connected.

(Hence ϕ: P → M is a homeomorphism.)
Side product results: Earthquakes on ideal polygons

As a side product of our study, we prove a discrete version of Thurston’s Earthquake Theorem:

Theorem (Danciger-M.-Schlenker)

Let $p, p'$ be two ideal hyperbolic polygons, both with vertices $v_1, \ldots, v_n$. There is a unique measured lamination $\lambda$ on $p$ so that the image of $p$ by the left earthquake along $\lambda$ is $p'$.

Given a combinatoric $\Gamma$ with a Hamiltonian cycle $\gamma$ such that $\theta \in A_{\gamma, \Gamma}$, we define $E_l(\theta_+)\circ E_l(\theta_-) \colon \mathcal{P} \to \mathcal{P}$ has a unique fix point.