Polyhedra inscribed in a hyperboloid and anti-de Sitter geometry.

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Historical introduction

Question (Steiner (1832))

What are the graphs obtained as 1*-skeletons of a convex polyhedron in* $\mathbb{R}^3$*?*

Theorem (Steinitz (1916))

An embedded graph in the sphere S^2 *is the* 1*-skeleton of a convex polyhedron in* $\mathbb{R}^3$ *if and only if it is 3-connected (that is, suppressing* 2 *vertices leaves a connected graph).*

Question (Steiner (1832))

Which ones are obtained from polyhedra inscribed in S^2*?*

Polyhedra inscribed in a sphere

Theorem (Steinitz (1927))

Some of those combinatorics cannot be realized by polyhedra inscribed in a sphere.

Theorem (Hodgson-Rivin-Smith (1992))

The answer depends on the existence of a solution to a set of linear equations and inequalities. (It can be decided in polynomial time.)

Question

What about polyhedra inscribed in a hyperboloid?

Our results

Let Γ be a graph embedded in S^2, we call $\mathcal{S}_\Gamma$ [resp. $\mathcal{H}_\Gamma$] the space of convex polyhedra inscribed in the sphere [resp. in the hyperboloid] with 1-skeleton Γ, up to projective transformations leaving the sphere [resp. hyperboloid] invariant.

Theorem (Danciger-M.- Schlenker)

$\mathcal{H}_\Gamma \neq \emptyset \iff \mathcal{S}_\Gamma \neq \emptyset$ *and* Γ *admits a Hamiltonian cycle.*

Theorem (Danciger-M.- Schlenker)

(i) *If* $\mathcal{H}_\Gamma \neq \emptyset$, *then* $\mathcal{S}_\Gamma \neq \emptyset$.

(ii) *If* $\mathcal{S}_\Gamma \neq \emptyset$, *then* $\{$*c.c. of* $\mathcal{H}_\Gamma\} \longleftrightarrow \{$*Hamiltonian cycles in* $\Gamma\}$.

Dihedral angles

Let Γ be a graph embedded in S^2, and let Γ^* be the graph dual to Γ.

Theorem (Rivin (1992))

Let $\theta\colon \Gamma^* \longrightarrow (0, \pi)$. There is a non-planar convex ideal polyhedron in $\mathbb{H}^3$ with combinatorics given by Γ and exterior dihedral angles given by θ if and only if:

(i) *For any simple closed curve c in Γ^* bounding a face, the sum of the values of θ on the edges of c is 2π.*

(ii) *For any simple closed curve c in Γ^* not bounding a face, the sum of the values of θ on the edges of c is $> 2\pi$.*

Rivin extended a result proved by Andreev (1970) for compact and ideal polyhedra P of finite volume with dihedral angles $\leq \pi/2$.

Induced metrics

Theorem (Rivin (1992))

Any complete hyperbolic metric of finite area on the sphere minus N points, with $N \geq 3$, is induced on a unique ideal hyperbolic polyhedron.

Rivin extended a result proved by Alexandrov (1944-50) for compact polyhedra.

Anti-de Sitter geometry

The *Anti-de Sitter space* $\mathbb{A}\mathrm{d}\mathbb{S}^3$ is

$$\mathbb{A}\mathrm{d}\mathbb{S}^3 = \{x \in \mathbb{R}^{2,2} : \langle x, x \rangle_{2,2} < 0\}/ \sim ,$$

where $x \sim y \iff \exists \lambda \in \mathbb{R}_+$ such that $x = \lambda y$,
with the induced Lorentzian metric. Its isometry group is $\mathrm{PO}(2,2)$.
The *ideal boundary* is

$$\partial_\infty \mathbb{A}\mathrm{d}\mathbb{S}^3 = \{x \in \mathbb{R}^{2,2} : \langle x, x \rangle_{2,2} = 0\}/ \sim \; .$$

A *convex ideal AdS polyhedron* is a convex polyhedron in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with its vertices on the ideal boundary. The faces of an ideal polyhedron are space-like.
The dihedral angles along the edges of the equator (called *exterior*) are in $(-\infty, 0)$, while the other dihedral angles lie in $(0, +\infty)$.

Dihedral angles

Let Γ^* be the graph dual to Γ.

Theorem (Danciger-M.- Schlenker)

Let $\theta\colon \Gamma^* \longrightarrow \mathbb{R}_{\neq 0}$. There is a non-planar convex ideal $\mathbb{AdS}$ polyhedron with combinatorics given by Γ and exterior dihedral angles given by θ if and only if:

(i) *The edges of Γ on which $\theta < 0$ form a Hamiltonian cycle in Γ.*

(ii) *For any simple closed curve c in Γ^* bounding a face, the sum of the values of θ on the edges of c is zero.*

(iii) *For any simple closed curve c in Γ^* not bounding a face, and containing at most two edges where $\theta < 0$, the sum of the values of θ on the edges of c is positive.*

Induced metrics

Theorem (Danciger-M.- Schlenker)

Let m be a finite-volume hyperbolic metric on S^2 with N cusps, and let e be a closed path going through each vertex exactly once. Then there is a unique ideal polyhedron $P \subset \mathbb{A}\mathrm{d}\mathbb{S}^3$ (up to global isometry) so that the induced metric on P is isometric to m and its path of external edges is homotopic to e.

Proof of $\Longleftarrow$

Now we will prove the following:

Theorem (Danciger-M.- Schlenker)

$\mathcal{H}_\Gamma \neq \emptyset \iff \mathcal{S}_\Gamma \neq \emptyset$ *and* Γ *admits a Hamiltonian cycle.*

Let $P \in \mathcal{S}_\Gamma$ and let γ be an Hamiltonian cycle for Γ. Let $\theta : \Gamma_1 \longrightarrow (0, \pi)$ be its dihedral angles which satisfies the conditions of Rivin's theorem on dihedral angles. Then we can define $\theta' : \Gamma_1 \longrightarrow \mathbb{R}_{\neq 0}$ by

- $\theta'(e) = \theta(e)$ if is not an edge of γ,
- $\theta'(e) = \theta(e) - \pi$ if is an edge of γ.

Then $\theta' : \Gamma_1 \longrightarrow \mathbb{R}$ satisfies the conditions of our theorem on dihedral angles. Therefore $\mathcal{H}_\Gamma \neq \emptyset$.

Proof of $\Longrightarrow$

Let $P \in \mathcal{H}_\Gamma$. Let $\theta : \Gamma_1 \longrightarrow \mathbb{R}_{\neq 0}$ be its dihedral angles, and let γ be the cycle of its exterior edges.
We can choose $t > 0$ such that:

- $\forall e \in \Gamma_1$ of Γ, $t\theta(e) \in (-\pi, \pi)$;
- $\forall$ s. c. c. c in Γ^* not bounding a face, and intersecting γ in k points, then the sum of the values of $t\theta$ on the edges of c is $> (2-k)\pi$.

Moreover, $t\theta(e) < 0 \iff e$ is an edge of γ.
Let $\theta' : \Gamma_1 \longrightarrow (0, \pi)$ be defined by:

- $\theta'(e) = t\theta(e)$ if is not an edge of γ,
- $\theta'(e) = \pi + t\theta(e)$ if is an edge of γ.

Then $\theta' : \Gamma_1 \longrightarrow (0, \pi)$ satisfies the conditions of Rivin's theorem on dihedral angles. Therefore $\mathcal{S}_\Gamma \neq \emptyset$.

Definitions

Let Γ be a 3-connected graph embedded in S^2, and let γ be an simple closed curve in Γ going through each vertex.

- We call $\mathcal{A}_{\Gamma,\gamma}$ the space of maps $\theta : \Gamma_1 \to \mathbb{R}$ such that:
  - for all $e \in \Gamma$, $\theta(e) < 0$ if e is in γ, $\theta(e) > 0$ otherwise,
  - the sum of the values of θ on the boundary of any face of Γ^* is zero,
  - the sum of the values of θ on any other cycle in Γ^* intersecting γ at most twice is positive.
- We denote by $\mathcal{P}_{\Gamma,\gamma}$ the space of polyhedral embeddings of S^2 in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with image an ideal polyhedron with 1-skeleton Γ such that the cycle of exterior edges is γ.
- The map $\Psi_{\Gamma,\gamma} : \mathcal{P}_{\Gamma,\gamma} \longrightarrow \mathcal{A}_{\Gamma,\gamma}$ sends a polyhedron to its exterior dihedral angles.

Sketch of the proof

Lemma (Danciger-M.- Schlenker)

$\Psi_{\Gamma,\gamma}$ is a proper local homeomorphism.
(Hence $\Psi_{\Gamma,\gamma} : \mathcal{P}_{\Gamma,\gamma} \longrightarrow \mathcal{A}_{\Gamma,\gamma}$ is a covering.)

Lemma (Danciger-M.- Schlenker)

1. *As $t \longrightarrow 0$, P_t converges to a flat polyhedron P_0.*
2. *For any Γ and γ such that $\mathcal{A}_{\gamma,\Gamma} \neq \emptyset$, $\exists$ a nbhd U of $\mathcal{P}^0_{\gamma,\gamma}$ and a nbhd V of 0 in $\mathcal{A}_{\gamma,\Gamma}$ s. t. $\Psi_{\gamma,\Gamma|U} : U \longrightarrow V$ is a homeomorphism.*

- $\exists$ a nbhd U of $\mathcal{P}^0_{\gamma,\gamma}$ in $\mathcal{P}_{\gamma,\Gamma}$ and a nbhd V of 0 in $\mathcal{A}_{\gamma,\Gamma}$ s. t. $\forall\theta \in V$ has a unique inverse image in U by $\Psi_{\gamma,\Gamma}$.
- Any $\theta \in V$ can have inverse images only in U.

(Hence $\Psi_{\Gamma,\gamma} : \mathcal{P}_{\Gamma,\gamma} \longrightarrow \mathcal{A}_{\Gamma,\gamma}$ has degree one, so is a homeo.)

End

Definitions

- By $\mathcal{P}_\gamma$ the space of ideal polyhedral embeddings of S^2 in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with vertices exactly at the v_i, with γ isotopic to the (oriented) equator, considered up to global isometry.
- By $\mathcal{M}_\gamma$ the space of complete hyperbolic metrics on the sphere S^2 with cusps at the v_i, marked by the closed curve γ, considered up to isotopy.
- By $\Phi_\gamma : \mathcal{P}_\gamma \longrightarrow \mathcal{M}_\gamma$ the map sending an ideal $\mathbb{A}\mathrm{d}\mathbb{S}$ polyhedral embedding to its induced metric.

Sketch of the proof

Lemma (Danciger-M.- Schlenker)

Φ_γ *is a proper local homeomorphism.*
(Hence $\Phi\colon \mathcal{P} \longrightarrow \mathcal{M}$ *is a covering.)*

Lemma (Danciger-M.- Schlenker)

$\mathcal{P}_\gamma$ *is connected, and* $\mathcal{M}_\gamma$ *is connected and simply connected.*
(Hence $\Phi\colon \mathcal{P} \longrightarrow \mathcal{M}$ *is a homeomorphism.)*

Side product results: Earthquakes on ideal polygons

As a a side product of our study, we prove a discrete version of Thurston's Earthquake Theorem:

Theorem (Danciger-M.- Schlenker)

Let p, p' be two ideal hyperbolic polygons, both with vertices $v_1, \cdots, v_n$. There is a unique measured lamination λ on p so that the image of p by the left earthquake along λ is p'.

Given a combinatoric Γ with a Hamiltonian cycle γ such that $\theta \in \mathcal{A}_{\gamma,\Gamma}$, we define $E_l(\theta_+)$ a left earthquake along the internal top edges of Γ. Note that this acts on the space of ideal hyperbolic polygons $\mathcal{P}$.

Theorem (Danciger-M.- Schlenker)

Let $\theta \in \mathcal{A}_{\gamma,\Gamma}$, then $E_l(\theta_+) \circ E_l(\theta_-) \colon \mathcal{P} \longrightarrow \mathcal{P}$ has a unique fix point.