LECTURE 1-2: Hyperbolic 3-mfds (Kahn-Funke)
LECTURE 3: Anti-de Sitter 3-mfds (VHHC a "AdS pF")
LECTURE 4: Analogues in higher rank case (Kahn-Hitchin)

MOTIVATION • hyp 3-mfds & Kleinian gps:
→ Origin in work of Paincare & Klein
→ Connections w/ topology thanks to Thurston's geometrization The
→ Complex analysis • Ahlfors-Beurling quasi-conformal theory
→ Complex dynamics — Thurston-Sullivan —

See PDF

There are spaces of hyp 3-mfds of fixed 3-mfld \( M \) is studied by

\( \text{AH}(M) = \{ \text{hyp 3-mfds with } \text{h.c. to } M \} \)

In the re-volume case the def space has non-trivial fiber

In the hyperbolic case the space has uniform fiber

The fiber has been fairly understood (Ahlfors, Bers, Gehring, Masur, Sullivan —)

but the \( \mathcal{G} \) is still somehow more "mysterious". We will consider

We consider \( M = S^3(0,1) \).

• Men discovered an analog of this theory in \( \text{AdS}^3 = \text{"Causal" } M^3 \)

3-dim AdS mfld has been studied so General Relativity toy model of gravity:

→ solutions of Einstein's equation w/ negative cosmological

constant but w/o matter. To add reality add some singularities along

- Hawking lines ("infinite particle" (if \( \text{area} < \pi \)) BSS6, BS8, CDU)
- Penrose (\( \mathcal{G} \) singularities along space-time lines)
- Cauchy lines (\( \mathcal{G} \) singularities along space-time lines)

BTZ black holes (space-like lines w/ part w/o future) &

Exterior BTZ black holes

There is an analog of this theory in the case of reps \( \pi_1 M \rightarrow G \) where

\( G \) is a higher rank real reductive. A branch often called \text{Higher Teichmüller Theory}. The role of Fuchsian reps is replaced by \( \text{Lie} \text{dim} \) reps & we...
1. $H^3$

- There are different models:
  - Upper-half space $(H^2, PSL_2(R))$ (Unit, Poincaré disk/Ball)
  - Hyperboloid (Projective in Klein model)

- Classification of elements in $PSL_2(R)$
  - Parabolic $\Rightarrow$ one fixed point in $\mathbb{R}H \Rightarrow \mathbb{T}^2A = 4$
  - Elliptic $\Rightarrow$ two fixed points in $\mathbb{H} \Rightarrow \mathbb{T}^2A \in \mathbb{Z}_{0,4}$
  - Hyperbolic/loxodromic $\Rightarrow 3$ fixed points in $\mathbb{H}$ (and no fixed points in $\mathbb{H}$)

2. Fuchsian/Kleinian groups

   - $G \subset PSL_2(R)$ is called Fuchsian [resp. Kleinian]
   - $G$ Fuchsian [resp. Kleinian] $\Leftrightarrow \mathbb{T}^2A = 4$ (for $H^3$)
   - Prop disc.

See [exercise] for equivalent formulations.

In fact, every such $G$ is obtained in this way.

In general it is hard to test discreteness, but a useful test is

- Theorem (Johannson's theorem) $A, B \in SL_2(C)$. If $<A, B>$ is Kleinian and non-elementary
  $\Rightarrow |\mathbb{T}^2A - 4| + |\mathbb{T}^2[A, B]| > 1$ (you can prove this: Shimizu's theorem)

- Def. $G$ elementary if it is virtually Abelian (i.e., $\mathbb{H} < P, G$)

Prop. $G$ elementary $\Rightarrow \exists$ a finite orbit for its action on $CP^1 = \mathbb{H}^3$

All Fuchsian-free elementary Kleinian groups are

i) $<g>$ $g$ parabolic except

ii) $<g, h>$ $g, h$ parabolic s.t. $\text{Fix}(g) = \text{Fix}(h)$
- What about the action of \( \Gamma \) Kleinian on \( \mathbb{C}P^1 \)?

\[ \Lambda(\Gamma) = \text{accumulation pts for } \Gamma \text{ in } \mathbb{C}P^1 \]  
**LIMIT SET**

\[ \Omega(\Gamma) = \text{max open subset of } \mathbb{C}P^1 \text{ where } \Gamma \text{ p.d.} \]  
**DOMAIN OF DISCONTINUITY**

- See exercises for further properties

AHUAS' MEASURE CONJ. Ahuas' company unifying Thurston-Brown, AHA, CANDY-SAM:

\[ \Lambda(\Gamma) \neq \mathbb{C}P^1 \Rightarrow \Lambda(\Gamma) \text{ has Lebesgue @ } \mathbb{C} \]

AHUAS' FINITENESS THM: \( \Gamma \) non-elem. \( \Rightarrow \) if Kleinian \( \Rightarrow \Omega/\Gamma \) is a finite union of R.S. of finite type \( \omega \times X < 0 \)

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**FUNDAMENTAL DOMAINS**

**DEF** A F.D. for \( \Gamma \subset \mathbb{H}^3 \) is \( F \subset \mathbb{H}^3 \) st:

1. \( \Gamma \cdot F = \mathbb{H}^3 \)
2. \( g(F \cap \partial F) = 0 \times \mathbb{R}^2 \)

**(Conchuent Domain)** \( F \times x \in \mathbb{H}^3 \) st: \( g \in F \) \( \Rightarrow \) \( g \times x \)

**Let** \( H_{0} = \{ y \in \mathbb{H}^3 / d(y, x) < d(y, g(x)) \} \) \( \Rightarrow \) \( D_{x} = \bigcup_{g \in \Gamma} H_{g} \)

- \( D_{x} \) is connected & locally finite \( (\Rightarrow \) \( \Gamma \subset \mathbb{C} \mathbb{H}^{3} \) isf \( \Rightarrow \) only)

- \( \Gamma \) is generated by sin\( \times \) parings, finitely many \( \Gamma(\Omega) \)

**CONCL** If for some \( \Omega \)\( D_{x} \) is finitely many \( \Rightarrow \) \( \Gamma \) is f.g.

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**SO** given \( \Gamma \), \( D_{x} \) is f.d.  Q: What about the converse?

**Poincaré Polyhedron Thm** (dim 2) \( \Rightarrow \) dim 3: got \( \Omega \) for which \( D_{x} \) has finitely many

Given \( \Gamma \) & angle pairing satisfying certain conditions (suff. nec.),

we can construct \( \Omega \) discrete (in dim 2 & 3)

**Examples**

1. \( \Gamma_{12} \) generated by \( (\alpha) \), \( (\beta) \)

2. \( \Gamma_{23} \) generated by \( (\beta) \), \( (\gamma) \)

3. \( \Gamma_{34} \) generated by \( (\alpha) \), \( (\beta) \)

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**Diagram**
\* G non-elementary, then

1) \( \Lambda \) is uncountably infinite, dense, unbounded (i.e. the G-orbit of any pt in \( \Lambda \) is dense in \( \Lambda \))

2) \( \Lambda \) is the closure of closed orbits with finitely many pre-images (\( \Lambda \) equals the closure of periodic orbits)

3) \( H \leq G \) if \( \text{ker} H \) is finite index in \( H \) \( \Delta G \Rightarrow \Lambda(H) = \Lambda(G) \)

4) \( \Lambda(G) \neq \emptyset \Rightarrow \text{int}(\Lambda) = \emptyset \) (Iff for all \( x \) in \( \Lambda\), \( x \) is periodic)

\* Suppose \( G \). Fix \( \Omega \neq \emptyset \), then

1) \( \Omega \) has either 1, 2 or infinitely many c.c.

2) Each c.c. of \( \Omega \) is either S.C. or infinitely connected

3) \( \Omega \) is G-invariant c.c. \( \Omega_1 \cup \Omega_2 \Rightarrow \text{each one is S.C.} \)

4) If \( \Omega \) are G-invariant c.c. \( \Omega_0 \), then all other c.c. are S.C.

**THM (Bowen)**: \( G \neq F \), not \( F \Rightarrow \text{dim}_{\text{haus}}(\Lambda) > 1 \).
Given a Fuchsian group $\Gamma \leq \text{PSL}_2 \mathbb{R}$, $\Gamma \cong \pi \Sigma$, what is $\Gamma \cap \mathbb{H}^3 \& \Omega$?

\[ \Lambda = \mathbb{R}P^1 \subset \mathbb{P}^1 \quad M_\rho \cong H^3 / \rho \cong \Sigma \times (0,1) \]

\[ \Omega = H^+ \cup H^- \quad M_\rho = (H^+ \cup \Omega) / \rho \cong \Sigma \times [0,1) \]

4. Classical Schottky $G_\rho$: Take $\kappa \geq 2$ pairs of mutually disjoint circles in $\mathbb{R}P^1 \setminus \{\infty\}$, $C_1, C_1', \ldots, C_\kappa, C_\kappa'$ with mutually disjoint interiors, and let $A_i \in \text{PSL}_2 \mathbb{C}$ s.t. $\{A_i(C_i) = C_i'\}

\[ \Gamma = \langle A_i \rangle \text{ is called a classical Schottky group} \]

pling long lemma $\Gamma \cong \mathbb{F}_\kappa$

5. What is $\Lambda$? What is $\Omega$? CANTOR SET INFINITELY CONNECTED

\[ \Omega / \rho \cong \Sigma \mathbb{R} \quad M_\rho \cong H^3 \text{ handlebody} \]

THE (HASKIER THEOREM 1) $M_\rho$ handlebody

2. $\Gamma$ is Schottky

3. $\Gamma$ is free & purely parabolic

3. Quasi-Fuchsian group

A Kleinian gp is QUASI-FUCHSIAN if $\Lambda(\Gamma)$ is a Jordan curve.

PROP: A Fuchsian $\Gamma$ is quasi-Fuchsian. TFAE

1. $\Gamma$ quasi-Fuchsian

2. $\Omega(\Gamma)$ has precisely two components

3. $\Gamma$ is a quasi-conformal deformation of a Fuchsian gp

$\exists \; \Gamma : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ quasi-conformal to $\Gamma$ quasi-conformal

$\exists F, G : \mathbb{C} \to \mathbb{C}$ or diffeo is K-quasi conformal if it sends

infinitesimally small round circle into infin. small ellipse whose ratio of semi-major axis length to semi-minor axis length is less than K.
To such a map, we can associate a **Bernoulli** measure \( \mu \) with \( \mu := \frac{\partial^2}{\partial z^2} \) for \( \mu = \mu(z) \)

\[
\phi: X \to Y \text{ s.t. } \phi \circ \theta = \theta \circ \phi
\]

\[
\mathbf{MT} (M=N \text{ simultaneous uniformization}) \quad \phi = \mathbb{C}(\Sigma) \times \Sigma
\]

\[
\text{Note: } \mu \text{ depends holomorphically on } \mu.
\]

\[
\mu(\Sigma) = \{ z \in \Sigma | \mu(z) = k < 1 \}
\]
A Beltrami differential is a differential locally written as $\mu(x) \frac{dx}{dt}$ with $\mu(x) \in L^\infty$ and $\|\mu\|_{L^\infty} < 1$. Think of $\mu$ as an infinitesimal elliptic field set of bold eccentricity where an ellipse has minor axis tilted at angle $\frac{\pi}{2}$ and eccentricity \( \frac{1 - \left| \mu(x) \right|}{1 + \left| \mu(x) \right|} \).

A B.D. of a QC map $f$ is given by local wave $\frac{\partial f}{\partial t} = f_0 \frac{dx}{dt}$ if will "straighten out", $\mu = 0$ to an infinitesimal circle field in the image $f$ conformal $\iff$ $\mu = 0$.

Amazingly the inverse pb can be solved! Next!
**Geometry of Convex Cores: Cocompact & Pleated Surfaces**

Let $L$ be a hyperbolic surface. A geodesic lamination $L \subseteq S$ is a closed set of disjoint simple closed geodesics (each one is called a leaf).

This definition can be made independent of the choice of the metric.

**Examples**
1. Milnor's $r$-p.c. example.
2. Geodesic spiralling around a s.c.c. (PDF)

**DEF** A transverse measure $\mu$ on $L$ is an assignment of a measure to each end $e$ to the leaves $e$ of $L$ that is invariant under pushforward map along the leaves of $L$ (aka: piecewise good surface).

**DEF** (Thurston 80s) A pleated surface in a hyperbolic 3-manifold $M$ is an isometry $f: S \rightarrow M$ from a surface $S$ which is totally geodesic almost everywhere and where the locus of points where it fails to be $S$ is every point has either a collar which is mapped totally geodesic or it is on a leaf of a geodesic lamination $L$ whose leaves are mapped totally geodesically.

If $f$ is convex cocompact (i.e., always bend in the same direction) then the amount of bending defines a measure on $M$.

**Thurston called these Uncramped Surfaces**

**DEF** The convex core $C_h$ of a hyperbolic 3-manifold $M$ is the smallest convex subset of $M$ whose inclusion is a homotopy equivalence.

**Thm (Thurston)** The boundary of the convex core of a hyperbolic 3-manifold $M$ is homeomorphic to the boundary of a locally convex pleated surface.

**Bonding Conv (Thurston)** $QF(\Sigma) \cong \{ (\lambda^+, \lambda^-) \in ML(\Sigma) \}$

**Thm (Bonahon-Otal)** The map is surjective.

**Convex Hull Conj (Thurston)** $QF(\Sigma) \cong \check{\Sigma} \times \check{\Sigma}$

$\Delta^+ \cong \Sigma \times \Sigma$, $\Delta^- \cong \Gamma \times \Gamma$
\[ \text{Thm (Smyth): } \exists \text{ Koebe homeo } \Omega \rightarrow B(0) \quad (\Rightarrow d_{\text{Teich}}(\partial \Omega, \partial B) \leq K) \]

\[ \text{Marden's theorem: } K \neq 2 \quad \text{best bound for } K \quad (Brock-Casson-C留元-Yang) \]

\[ \text{Thm (Ahlfors): } \Omega \text{ s.c. } \gamma \text{ geodesic } \quad \Omega \backslash \gamma \text{ represented by } \alpha \quad \text{Then} \quad |K| < 1 \]

\[ \text{Thm (Labourie): Given } g \in \text{OF}(\Sigma), \text{ the complement of } C_g \text{ admits a foliation by } \kappa \text{-surfaces } \omega / \kappa \in (-1,0) \]

\[ \text{A } \Sigma \in \text{ M}, \text{ Mhyp } 3 \text{-mfld is a } \kappa \text{-surface if it has constant negative Gaussian curvature } = \kappa. \]

\[ \text{con} \quad \text{Given } \kappa, \kappa < 0 \quad \Rightarrow (X^+, X^-) \in \mathbb{C}(\Sigma) \times \mathbb{C}(\Sigma) \quad \exists ! \quad \rho \in \text{OF}(\Sigma) \]

\[ \text{st. } (X^+, X^-) = (X^+, X^-) \]

\[ \text{Thm (Labourie): } \exists \text{ holds} \]

\[ \text{Thm (Scott-Camen): } \exists \text{ holds} \]

\[ \text{My * similar statement where you perceive III fundamental form (instead of I fundamental) which correspond to B"ohring and} \]

\[ \text{* Also a universal version of these thin convex for } \]

\[ \text{given circles not invariant under } \pi, S. \]

\[ \text{B Geometry of cusps:} \]

\[ x \in \text{ M}, \text{ intrinsic mod. } \text{inj}^+_x(H) = \text{upf } \text{ from } \text{Bx}(x) \text{ is embedded} \]

\[ \text{e} \quad \text{cusp part of } H = \{ x \in M1 \text{ inj}_x(H) < e \} \]

\[ \text{HANNDJUZU M 2-a 3-dim hyp mfld, } \exists E_0 > 0. \text{ If } E < E_0 \text{ then} \]

\[ \text{Me counts of i) toroidal collar around short geodesics (torus tubing)} \]

\[ \text{ii) handle ribbon of cusps} \]

\[ \mathcal{P} = \text{ Kleinian gp counting of paraboles w/ common fixed pt } \pi \]

\[ \Rightarrow \mathcal{P} \equiv \mathbb{Z}^2 \text{ (quasi-isom)} \quad \mathcal{P} \equiv (1,0) \quad (1,1) \]

\[ \Rightarrow N_{\mathcal{P}} \equiv \text{ SL(2, } \mathbb{Z} \text{) w/ missing core components} \]

\[ \mathcal{P} \equiv \mathbb{Z} \quad \text{ (weaker)} \quad \mathcal{P} \equiv (1,0) \quad \Rightarrow \quad N_{\mathcal{P}} \equiv \text{ infinite petal cylinder w/ missing core components} \]

\[ \text{where } N_{\mathcal{P}} = \{ (t, 1) t > 3 \} \text{ homog of hyperb. } \]

\[ \text{The } \mathcal{P} \text{ Kleinian gp is GEOM finite if } \exists \text{ a finite covol F.D.} \]
\[ \frac{\Delta H^\circ (S)}{\Delta H^\circ (S)} \leq 0 \] 

So if \( \sigma \) is possible in \( \mathcal{O}_1 \), then it is possible in \( \mathcal{M}_3 \)

1. The converse is not True.

2. This gives obstructions to cellular groups in this setting.
The definition does not generalize well, but Bowditch proved that if many other (equivalent) definitions of GE exist.

**Thm (Bowditch)**

1. \( \Gamma \) is quasi-isometric (QI) if
   1. \( \Gamma \) is the union of a compact set \( K \) and a finite set of side-cusp regions.
   2. \( \Gamma \) consists entirely of cusps, limit points, and boundary points.
   3. \( \exists \) an oriented geodesic \( \gamma \) with limit points ending.
   4. \( \exists \) points \( p \) in \( \partial \mathcal{C} \) such that \( \partial \mathcal{C} \) is compact.

**Remark:**

1. \( \exists \) a bound on the number of every finite subgraph of \( \Gamma \).
2. \( \exists \) a compact set \( N \) such that \( \gamma \) has compact closure.
3. \( \exists \) a non-empty, closed, convex, compact subset \( C \) of \( \mathcal{C} \) such that \( C \) is compact.

**Question:** How can you construct these? Use a limit construction or plumbing construction?

**Proposition (Kingsley)**

1. \( \exists \) Kleinian group \( \Gamma \) which are not QI.

2. A cusp group in which \( \Gamma \) is QI, where \( \mathbb{H} = \mathbb{C} \) (where \( \mathbb{H} = \mathbb{C} \)) and \( \mathbb{R} \) is QI.

3. Given \( X_\infty \in \mathcal{C}(S) \), then \( \exists \) \( X_\infty \rightarrow X_\infty' \in \mathcal{C}(S) \) is compact.

4. \( \exists \) \( c \) such that \( \gamma \) is a loop in \( \mathcal{C}(S) \).

**Thm (Thurston)**

1. Given \( X_\infty \in \mathcal{C}(S) \), then \( \exists \) \( X_\infty \rightarrow X_\infty' \in \mathcal{C}(S) \) is compact.

2. \( \exists \) \( \gamma \) such that \( \gamma \) is a loop in \( \mathcal{C}(S) \).

**Thm (Thurston, Double Limit Thm)**

1. Suppose \( q \in \mathcal{Q}(S) \) and \( \lim \mathcal{Q}(S) \) is compact.

2. \( \exists \) \( \gamma \) such that \( \gamma \) is a loop in \( \mathcal{C}(S) \).

**Example:**

- All \( c.c. \) of \( S \) are triangles or punctured bigons.
& support i \( (S, S') \neq \emptyset \) then \( \exists p_{\infty} \rightarrow \gamma \rightarrow \text{geometrize} \uparrow \uparrow \)
which is strongly degenerate & has unique laminar core \( S^+ \).

\[ t = \{(0,1)\} \]

Thurston: \( \mathbb{M} \) of a word \( N_i, j \) an equivalence class of a merical \( \mathbb{M} \) of semi-open subsets \( U_i \supset U_j \rightarrow \text{st.} \mathbb{M} U_i = \emptyset \).

\( (U_i) \sim (V_j) \) if \( \forall i, j, k \:
\text{st. } V_j \subset U_i \cap U_k \Rightarrow \emptyset ; \)

Any \( U_i \) is a motion of \( E \) \( \Rightarrow \exists \) no curves (on compact \( N_i \to N_i \))

\( E \) is geometric if it has a motion which does not intersect the convex core \( \mathcal{C} \).

\( E \) is **simply degenerate** (complete) if \( \exists \) motion completely contained in the convex core.

\[ \Rightarrow \mathbb{M} \] is **simply degenerate** if \( \exists \) \( a_i = \infty \subset S \) \( \geq \) good \( \text{rep.} \in \mathbb{M} \) are eventually contained in a motion of \( E \).

**Thurston:** \( \exists ! \mathcal{E}_E \in \text{UML}(S) \mathcal{E}_E \rightarrow 1_{\mathcal{E}_E} \in \text{UML} \cong \gamma_+ \text{ exist } E \)

\( 1_{\mathcal{E}_E} \) is filling \( \mathcal{E}_E = \text{lo}(S) \mathcal{E}_E \mathbb{E} = 1_{\gamma_+} \text{ exist } E \)

Bonahon (3H incomparable) \( N \subseteq \mathbb{M}(H) \Rightarrow \) every end is C. For S.D.

\( \text{UML}(S) \) has quotient topology for \( \text{H} \cup \gamma_+ \). NOT Hausdorff

\[ \mathcal{E}(S) = \{ \text{filling laminations} \} \] is Hausdorff

**Marden tameness con:** Every \( f. g. \) \( \mathbb{K} \). \( \gamma_+ \) is tame \((\mathcal{E} \subset \text{int cpt}) \)

**BSN density con:** (Minsky; Bowditch) Every \( f. g \) \( \mathbb{K} \). \( \gamma_+ \) is the alg. limit of \( \gamma_+ \).

**Thurston ending laminar con:** (Minsky; Bowditch-Canary-Minsky)

Every \( f. g \) \( \mathbb{K} \). \( \gamma_+ \) is determined up to hom, by \( \gamma_+ \)

"END INVARIANT"
\[ \text{Types of convergence} \quad (\mathbb{R}^n \rightarrow \mathbb{R}^n) \]

**Def:** \( P_n \rightarrow P \) **locally if** \( P_n(x) \rightarrow P(x) \quad \forall x \in \mathbb{R}^n \)

\( P_n, P \in \mathcal{G}_m \to H \) **Geometrically if**

i) \( \forall h \in H \exists \ g_m \in G_m \quad g_m \rightarrow h \)

ii) \( \forall g_{m_k} \in G_{m_k} \text{ s.t. } g_{m_k} \rightarrow h \in \mathbb{R}^n \to h \in H \).

**Theorem:** Geometric convergence is equivalent to polynomial convex.

For \( \mathbb{R}^n \) - manifolds.

**Theorem (Weil-Petersson)** \( G_m \subset \mathbb{R}^n \) & \( \bar{\Omega} \)-complete \( \Rightarrow \Lambda_m \rightarrow \Lambda_H \) in holomorphically convergent of closed subset of \( G \).

**Theorem (Cohn-Keown):** For \( \mathbb{R}^n \) - manifold, \( P_n \rightarrow G_m \) \( \mathcal{G}_m \to H \) convergent, \( P_n \rightarrow G_m \) \( \mathcal{G}_m \to G_m \) \( \mathcal{G}_m \to G_0 \) \( \mathcal{G}_m \to G_0 \) is Kleinian, manifold, \( \mathcal{G}_m \to G_m \) \( \mathcal{G}_m \to G_0 \) \( \mathcal{G}_m \to H \).

**Example (Jönsson):** Let \( P_n : \mathbb{C} \rightarrow \mathbb{R}^n \) defined by

\[ P_n(z) = \begin{pmatrix} \exp(\omega_n) & \sinh(\omega_n) \\ 0 & \exp(-\omega_n) \end{pmatrix} \]

where \( \omega_n = \frac{2i}{n^2 + \pi^2} \).

\[ P_n \rightarrow \bar{\Omega} \subset \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \]

\[ P_n \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

**Theorem:** \( P_n \rightarrow P \) \( \mathcal{G}_m \to \mathcal{G}_n \) Then the convergence is strong if

1. \( \exists \) new parameter \( \mathbb{R}^n \) is CF. \( \mathbb{H} \)

**Example:** Topology \( \mathbb{A}_n(\mathbb{R}) \).
Lecture III

Hyperboloid Model

1. AdS³ \overset{\text{EQUIVALENT MODEL}}{\leftarrow} \text{PROJECTIVE MODEL}

(PSL₂ \mathbb{R} \text{ MODEL})

Let \( \mathbb{R}^\mathbb{2,2} = (\mathbb{R}^4, \langle \cdot, \cdot \rangle) \) such that

\[ \langle x, y \rangle_{\mathbb{2,2}} = x_1 y_1 + x_2 y_2 - x_3 y_3 - x_4 y_4. \]

\[ q(x) = \langle x, x \rangle_{\mathbb{2,2}} \]

AdS³ = \{ x \in \mathbb{R}^4 | q(x) < 0 \}/\sim \text{where } \sim \text{ coming from } q.

It is endowed with three Lorentzian subgroups coming from \( q \).

You can define \( \hat{\text{AdS}}^3 = \{ x \in \mathbb{R}^4 | q(x) = -1 \} \) \( \overset{2:1}{\sim} \) \( \text{AdS}^3 \) & you endow \( \text{AdS}^3 \) with Lorentzian subgroups, so it is good (something).

\( \partial \text{AdS}^3 = \{ x \in \mathbb{R}^4 | q(x) = 0 \} / \mathbb{R}^+ \) Extends \( \partial \text{AdS}^3 \sim \mathbb{R} \mathbb{P}^1 \times \mathbb{R} \mathbb{P}^1 \)

& it coincides w/ the Veronese embedding \( \mathbb{R} \mathbb{P}^1 \rightarrow \mathbb{R} \mathbb{P}^3 \).

In the chart \( x_4 \neq 0 \) or \( x_3 \neq 0 \)

\[ \text{Isom} (\text{AdS}^3) = \text{PO}(2,2) \]

\[ \text{Isom}^+ (\text{AdS}^3) = \text{PSO}(2,2) \]

\[ \text{Isom}^{\pm, \mathbb{R}} (\text{AdS}^3) = \left[ \text{PO}(2,2) \right] \]

The double covering in \( \partial \text{AdS}^3 \)

\( \mathbb{R}^\mathbb{2,2} \) is Lorentzian.

\[ \text{Time-like} \quad <v, v> > 0 \]

\[ \text{Light-like} \quad <v, v> = 0 \]

\[ \text{Space-like} \quad <v, v> < 0 \]

\[ \text{Outward} \quad \Rightarrow \text{PO}^+ \quad \text{&} \quad \phi(x) > 0 \]

\[ \text{Light-like} \quad \text{if} \]

\[ \text{Degenerate} \quad \Rightarrow \text{PO}^+ \quad \phi(x) = 0 \]

Another description \( \text{AdS}^3 \sim \text{PSL}_2 \mathbb{R} \) via \( \phi: \mathbb{R}^{2,2} \rightarrow \mathbb{M}_2 \mathbb{R} \)

\[ (x, \ldots, x_4) \mapsto (-x_1, x_3, x_4, x_4) \]
1. Discrete Action of Discrete Subgroups

2. What can we say about the action \( \Gamma \bowtie \text{AdS}^3 \) where \( \Gamma \leq \text{Isom} (\text{AdS}^3) \)?

Let \( \Gamma = \text{PSL}_2 \mathbb{R} \) 
\( \Rightarrow \Gamma \bowtie \text{prop. disc.} \) on \( \text{AdS}^3 \). In fact
\( \text{M}_\text{p} = \text{AdS}^3 / \sim \cong T \mathbb{H} \) & it has a \( \text{PSL}_2 \mathbb{R} \)-ort.
\( \text{X}_p \) (Every \( \text{PSL}_2 \mathbb{R} \)-ort. can be interpreted as \( \text{AdS}^3 \)-ort.)
\( \text{PSL}_2 \mathbb{R} \) is a subgroup of \( \text{AdS}^3 \)
\( (\mathbb{C} \times \mathbb{R} \setminus \{0\}) / (\mathbb{R}^+ \times \mathbb{R}^+ \setminus \{1\}) \)

The analog of Mostow's Rigidity is not true for \( \text{AdS}^3 \).

2. Let \( \Delta = \text{PSL}_2 \mathbb{R} \) does not act prop. disc. on \( \text{AdS}^3 \)

\( \text{Example} \) Let \( \gamma \in \Gamma \) of infinite order, then \( \langle \gamma, \gamma \rangle \) is an infinite volume of \( \Delta \) which fixes \( \gamma \in \text{AdS}^3 \)

But there is a domain of discontinuity \( \Omega := \text{PSL}_2 \mathbb{R} \setminus \text{AdS}^3 \)
\( \Omega / \Delta \cong \mathbb{H} \times (0, 2\pi) \) & \( \mathbb{H} \times \{\pi\} \) is a totally \( \Gamma \)-invariant sphere-like surface

\( \Theta \rightarrow 0 \) when \( \Theta \rightarrow 0 \) or \( \Theta \rightarrow 2\pi \)

\( \text{AdS}^2 \) surface (aka \( \text{AdS}_2 \)-PF)

3. Globally Hyperbolic Maximal Compact \( \text{AdS}^2 \)-manifolds (aka \( \text{AdS}_2 \)-PF)

\( \text{Def} \) A 3-dim \( (\text{AdS}^2, \text{Iso} (\text{AdS}^2)) \)-manifold is called
- Globally Hyperbolic if \( 3 \)Conway surface (space-time surface intersecting each inextendable time-like geodesic once)
- Maximal if any isometric embedding \( \text{M} \rightarrow \text{M}' \), \( \text{M}' \subset \text{GH} \), is an isometry.
Given $M \in \text{Gr}(S)$ there is a 1 maximal extension.

$$M \text{ with } M \cong S \times \mathbb{R}$$

$$\text{Gr}(S) = \mathbb{R} \text{ has } \text{ AdS} \text{ metric} \text{ on } S \times \mathbb{R}^3 / \partial S$$

**Theorem (Hess)**

$$\text{Gr}(S) \cong \mathbb{R} \times \mathbb{R}$$

**Example**

Given $M \in \text{Gr}(S)$ there is a $\tilde{M} \rightarrow \text{AdS}^3$

$$p_0 = (p_1, p_2) : \tilde{M} \rightarrow \text{PSL}_2 \mathbb{R} \times \text{PSL}_2 \mathbb{R}$$

**Step 1:** $p_1, p_2$ are unique and full Will.

We will use **Coverman's criterion**.

$$\pi : S \rightarrow \text{PSL}_2 \mathbb{R} \text{ of } \phi$$

$$\implies \text{len}(p) = 1 \times \text{len}(E_p) = 1 \times (S) = 2p - 2 \text{ where } E_p = S \times \mathbb{R}^2$$

$$\phi : (x, s) = (g^x, p(x) \cdot s)$$

Where $p = \text{maximal } \phi_1 \text{ and } \phi_2$

We want to construct such $\phi_1 : T^S \rightarrow E_p$.

**Step 2:**

Given $\tilde{p}_0 \in \text{PSL}_2 \mathbb{R} \times \text{PSL}_2 \mathbb{R}$ we want to find an open domain $\tilde{\phi}$ such that $\tilde{\phi} \text{ is } \text{ AdS}^3 \text{ for } \tilde{p}_0 \text{ AdS}^3$

You also want to see a boundary surface in $\tilde{E}_p$.

- Given $p_1, p_2 \in \mathbb{R}$ let $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a $p$-extension.

$$\phi : T^S \rightarrow \tilde{S} \times \mathbb{R}^2$$

Now $\phi_1, \phi_2$ are isom. and equivalent.

$$\phi : T^S \rightarrow \tilde{S} \times \mathbb{R}^2$$

The graph $\text{Graph}(\phi) \subset \text{AdS}^3$ is $p$-equivariant.

$$\text{Conclude the vector will } \text{Gr}(\phi) : CH(\phi) = CH(\tilde{S}(\phi))$$
Because \( \Phi \) is O.P. \( \Rightarrow G(\Phi) \) is space-like \( \Rightarrow \mathcal{O}^+ \mathcal{C}^p \) is Cauchy

\[ \mathcal{O}(p) \in \mathcal{O}^+ \mathcal{C}^p \text{ if and only if } \mathcal{O}(p) \text{ is a timelike vector} \]

Use the Cauchy-Buneman-Geroch to find the conformal compactification.

**In fact,** \( G = \Lambda_p \) and \( \mathcal{O}(G) = \{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} \text{(and plane) is disjoint from } G \} \)

**Example:** The case \( p = p_\infty \) was the case discussed in Example 2.

\[ \text{Conf (mess)} \] \( p = \mathcal{O}(S) \Rightarrow \mathcal{O}^+ \mathcal{C}(p) \) is a bounded convex space of points.

In particular, if \( m^+, m^- \) are the metrics on \( \mathcal{O}^+ \mathcal{C}(p), \mathcal{O}^- \mathcal{C}(p) \) and \( \mathbf{X}^+, \mathbf{X}^- \) are the sending domains.

\[ E_x(\mathbf{X}) \quad m^+ \xrightarrow{H_x} E_x(\mathbf{X}) \quad m^- \]

\[ H_x \quad p \quad H_x \quad p \]

\[ E_x(\mathbf{X}) \]

**Euphuastic** **Anagrams**

**Define** twist \( \leftrightarrow \) extrema

**Euclid's** Theorem \( \Rightarrow \mathcal{O}(S) \text{ is convex} \)

Given \( \mathbf{x} \in \mathcal{O}(S) \), \( \mathbf{X} \in S \) s.t. \( \mathbf{x} = \mathbf{X} \cdot \mathbf{t} \)

\[ \mathbf{X} \in \mathcal{H}(S) \]

**An enlargeable** deformation \( E_x(\mathbf{X}) \) is the limit of \( D_{\mathbf{x}, t}, \) where \( (\mathbf{x}, t) \to \mathbf{x} \)

It is a very useful tool:

- Use it to prove Nielsen Realisation PB (every finite group can be realized as the ism of \( \mathcal{H}(S) \))
- \( S \times \mathbb{R} \) is \( \mathcal{H}(S) \)

**Thurston** \( \forall \mathbf{x}, \mathbf{x}' \in \mathcal{O}(S) \) \( \exists! \mathbf{X} \in \mathcal{H}(S) \) s.t. \( E_x(\mathbf{X}) = (\mathbf{x}^+, \mathbf{x}^-) \)

\[ \text{Conj (mess)} \] \( \forall \mathbf{x}, \mathbf{x}' \in \mathcal{O}(S) \)

**Thurston** \( \exists! \mathbf{X} \in \mathcal{H}(S) \) such that \( \mathbf{x}, \mathbf{x}' \in \mathcal{O}(S) \)

\[ \mathcal{O}(\mathbf{x}) \text{ holds} \]

(\( \Lambda_p, \Lambda_{\mathbf{x}} \text{ hold} \))
The two cases are infinitesimally (prob. globally) equivalent.

Given $p \in CH(\mathcal{S})$, the complement of $E_p$ in $\mathcal{D}(\Lambda_p)$ admits a solution by $k$-inf.  

where $k \in (-\infty, -1)$

Then (Borabot-Quesin-Zee) holds.

(see Touboul, extended) then result $p$ is one of nucleon.

not necessary of contour curvature...
IV. Quasi-Hitchin reps

**Higher Teichmüller Theory** for higher-rank semi-simple Lie gp (like $\text{PSL}(d,\mathbb{R})$) have an c.c. of $\mathcal{X}(\mathbb{H},\mathbb{G})$ st.

- They consist entirely of discrete & faithful reps.
- They have some of the nice features of Teichmüller (S).
- They parametrize some geometric...

**The first example of such a component was given by Hitchin 92.**

\[(\mathbb{P}^1,\text{projective line}) \text{ show that Hitchin component consists of d & f. reps.}\]

\[
\begin{align*}
\text{H}^+(S,\text{PSL}(d,\mathbb{R})) & \times \mathcal{X}(\mathbb{H},\text{PSL}(d,\mathbb{R})) & \text{if } d = 2k, \text{PSL}(2k) \text{ R}\n\text{H}^+(S,\text{PSL}(d,\mathbb{R})) & \times \mathcal{X}(\mathbb{H},\text{PSL}(d,\mathbb{R})) & \text{if } d = 2k + 1, \text{PSL}(2k+1) \text{ R}\n\end{align*}
\]

- Guichard-Wiedhaus (work in progress) are classifying all such c.c.

Today: we will discuss quasi-Hitchin reps.
\[ l_{s1} = \text{Sp}(2m, F) / P_i(2m, F) \]
\[ l_{s1} = \text{Sp}(2m, F) \]
\[ Q_i = \text{Bel} (p) \]
\[ p = \text{some in } F^{2m} \subseteq \text{R}^{2m} \]

**Proof (obviously):** \( p : \pi \Sigma \rightarrow \text{Sp}(2m, F) \) is \( Q_i \)-Anosov if \( \exists \) cts, \( p \) equiv.

\[ \Sigma^i : \Theta \rightarrow \text{R}P^1 \rightarrow \text{R}(F^{2m}) = l_{s1} (F^{2m}) \] s.t.

1. **Dynamics preserving:** \( \forall \Sigma \subseteq \text{R}(F^{2m}) \) \( \Sigma \) is \( p \) equiv.

2. **Transverse:** \( \forall \Sigma \subseteq \text{R}(F^{2m}) \) \( (p \Sigma ) = F^{2m} \)

3. **Contraction-expansion properties:** \( \forall x \in \Sigma \) \( \lambda_i (p(x, y)) \rightarrow 0 \)

\[ a = \{ \lambda_i \in \text{R} - 1 \} \quad \lambda_i \in \text{R} 3 \quad \text{c sp}(2m, F) = \text{Lie} (G) \]

\[ a^+ = \{ \lambda_i > 0 \} \quad \lambda_i \in \text{R} 3 \quad \text{c sp}(2m, F) = \text{Lie} (G) \]

\( G = k \exp (a^+ ) k \) is unique (CARTAN DECOMPOSITION)

\[ \mu : G \rightarrow a^+ \quad \forall x \in G \] \( a \in a^+ \), \( y = k \exp (a) \gamma_k \to a \)

\[ \text{UT } a_i = E_i - E_i \gamma_i \in a^* \quad a_i = 2 \gamma_i \in a^* \quad \epsilon_i \left( \begin{array}{c}
1 \\
-1
\end{array} \right) = \lambda_i 
\]

**Hitchin in \text{Sp}(2m, R)**

**Eq.** \( \pi, \Sigma \rightarrow \text{Sp}(2m, R) \) can be realized by:

- \( \text{Sp}(2m, R) \) contrasted.

- \( p(x, y) = p(x, y) ; x, y \in \text{R}^2 
\]

**There are Fuchsian reps in \text{Sp}(2m, R)**

**Hit (\Sigma, \text{Sp}(2m, R))** is the c.c. of \( \Sigma \in (\text{Sp}(2m, R)) \) containing Fuchsian reps.

**2. Quasi-Hitchin in \text{Sp}(2m, C):**

- \( \text{Fuchsian in } \text{Sp}(2m, C) \rightarrow \text{Sp}(2m, R) \)

- **Deform Fuchsian reps remaining \text{Q}_1 \text{-Anosov.**

**3. Other \text{Sp}(2m, C) embeddings, maximal reps positive...**

**Proposes (Labourie, GhN):** \( p \) \( Q_i \text{-Anosov } \rightarrow 

**A)** \( p \) is discrete & faithful

\[ p \ni 1 \rightarrow \text{C} \quad \sigma \rightarrow \text{C} \]...
Do quasi-Hitchin correspond to geun structures?

Given \( p : \Sigma \rightarrow G \), with Amosov \( S : \Theta_{p}(\Sigma) \rightarrow \mathbb{P}(\mathbb{C}^{2}) \).

Let \( K_{\Sigma} := \bigcup_{t \in \Theta_{p}(\Sigma)} K_{\Sigma}^{t} \) where \( \forall \leq \mathbb{P}(\mathbb{C}^{2}) \).

\( \Omega_{\Sigma} := \log(\mathbb{C}^{2}) \setminus K_{\Sigma} \).

The \( \Omega_{\Sigma} \) is a domain of discontinuity for \( p \triangleright \mathbb{P}(\mathbb{C}^{2}) \).

\( \mu = \text{open}, \mathbb{P}(\mathbb{C}^{2})-\text{improper domain where } \rho(\pi, \Sigma) \triangleright \text{open assoc. & compact} \).

What is the topology of \( \Omega_{\Sigma} / p \)?

\text{Conjugacy (Dumas-Sanders)} \( \Omega_{\Sigma} / p \) are fiber bundles over \( \Sigma \) with

\text{then a compact homogeneous symmetry space.}

\text{Dumas-Sanders} noted the \( SL_{2} \mathbb{C} \) -

\text{Theorem (Alessandri-Frenkel) Pf for } \text{Sp}_{4} \mathbb{C} \text{ dim(} \log(\mathbb{C}^{4}) \text{)} = 6 = \text{dim}(\mathbb{S}^{1})

Our theorem comes from the study of \( SL_{2} \mathbb{C} \)-orbits of \( \log(\mathbb{C}^{4}) \).

There are 3 \( SL_{2} \mathbb{C} \)-orbits in \( \log(\mathbb{C}^{4}) \):

\text{one open orbit} : \( \log(\mathbb{C}^{4}) \setminus K_{\Sigma}^{t} \text{ is ideal regular hyperbolic} \)

\text{one orbit not closed nor open} : \( K_{\Sigma}^{t} \setminus \mathbb{S}^{2}(\mathbb{P}^{1}) \text{ is degenerated} \)

\text{two closed orbit} \( \mathbb{S}^{2}(\mathbb{P}^{1}) \text{ are closed orbit} \).

\text{Veronese Embedding:}

\( \mathbb{S}^{1} : \mathbb{R}^{1} \rightarrow \mathbb{C}^{1} \text{ extends to } \mathbb{S}^{1} : \mathbb{C}^{1} \rightarrow \mathbb{C}^{2} \)

\( [a:b] \rightarrow [(ax-by)^{3}] \)

\( \mathbb{S}^{2} : \mathbb{R}^{1} \rightarrow \log(\mathbb{C}^{4}) \)

\( [a:b] \rightarrow (ax-by, (ax-by)^{3}) \)

\( r : \pi, \Sigma \rightarrow \text{Sp}(4, \mathbb{C}) \) Fuchsian mod \( \mathbb{S}^{1}, \mathbb{S}^{2} \) are defined by \( \mathbb{S}^{1}, \mathbb{S}^{2} \).
So we have \( \log(I^4) = \Omega_{5/4} \). It is \( \rho \)-equivalent by def.

The map is the

This is the map which proves that \( \Omega_{5/4} \) is a bundle over \( \Sigma \).

Q: What can we say about the fiber?

Open Q: Can we generalize this to other reps \( G \)?

Can we understand the quasihom (\( G \)?

of the symmetric space.

What can we say about limits of these reps?