The geometry of symplectic quasi-Hitchin representations.

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Fuchsian and quasi-Fuchsian theory

Every quasi-Fuchsian \( \rho: \pi_1(\Sigma) \longrightarrow \text{SL}(2, \mathbb{C}) \cong \text{Sp}(2, \mathbb{C}) \) acts p.d. on \( \mathbb{H}^3 \) and on \( \Omega_\rho \subset \partial \mathbb{H}^3 = \mathbb{C}P^1 \cong \text{Lag}(\mathbb{C}^2) \).

- \( M_\rho = \mathbb{H}^3 / \rho \cong \Sigma \times \mathbb{R} \);
- \( \Omega_\rho / \rho \cong \Sigma^+ \sqcup \Sigma^- \).

\textbf{Question}

What are ‘quasi-Fuchsian’ representations in \( \text{Sp}(4, \mathbb{C}) \)?

\textbf{Question}

What is the topology of the quotient \( \Omega_\rho / \rho \) for \( \Omega_\rho \subset \text{Lag}(\mathbb{C}^4) \)?
Symplectic spaces and group

- Symplectic space \((V_K, \omega_K)\) (with \(K = \mathbb{R}, \mathbb{C}\)).
- Symplectic group \(\text{Sp}(V_K, \omega_K)\).
- \(L \subset V_K\) isotropic if \(L \subset L^\perp_{\omega_K}\) and Lagrangian if \(L = L^\perp_{\omega_K}\).

**Example**

- \(V_\mathbb{C} = \mathbb{C}^4 = \mathbb{C}^{(3)}[X, Y]\) and \(\omega_\mathbb{C}\) defined by \(\omega_\mathbb{C}(X^3, Y^3) = 1\) and \(\omega_\mathbb{C}(X^2 Y, XY^2) = -\frac{1}{3}\).
- \(\text{Sp}(V_\mathbb{C}, \omega_\mathbb{C}) \cong \text{Sp}(4, \mathbb{C})\).
Symplectic Anosov representations

Definition (Labourie)

\[ \rho: \pi_1(\Sigma) \longrightarrow \text{Sp}(2n, \mathbb{K}) \text{ is } Q_1-\text{Anosov} \text{ if } \exists \text{ continuous } \rho-\text{equivariant } \xi_1^{\rho}: \partial_\infty(\pi_1(\Sigma)) \longrightarrow \mathbb{P}(\mathbb{K}^{2n}) \text{ s.t.} \]

1. \( \xi_1^{\rho} \) is dynamics preserving \( (\forall \gamma \in \pi_1(\Sigma), \xi_1^{\rho}(\gamma^\pm) = \rho(\gamma)^\pm) \);
2. \( \xi_1^{\rho} \) transverse \( (\forall t \neq s, \xi_1^{\rho}(t) \text{ and } \xi_1^{\rho}(s) \text{ are transverse}) \);
3. + contraction/expansion properties.

Example

- **Hitchin reps**: conn. comp. of \( \mathcal{X}(\pi_1(\Sigma), \text{Sp}(2n, \mathbb{R})) \) containing
  
  *Fuchsian reps* \( \pi_1(\Sigma) \xrightarrow{\text{d. f.}} \text{SL}(2, \mathbb{R}) \xrightarrow{\text{irred.}} \text{Sp}(2n, \mathbb{R}). \)

- **\( Q_1 \)-quasi-Hitchin reps**: deformation of Hitchin reps
  
  \( \pi_1(\Sigma) \xrightarrow{\text{d. f.}} \text{SL}(2, \mathbb{R}) \xrightarrow{\text{irred.}} \text{Sp}(2n, \mathbb{R}) \longrightarrow \text{Sp}(2n, \mathbb{C}) \) inside
  
  \( Q_1 \)-Anosov reps \( \mathcal{X}_{Q_1}(\pi_1(\Sigma), \text{Sp}(2n, \mathbb{C})). \)

- def. of \( \text{SL}(2, \mathbb{R}) \) embeddings; **Maximal reps**; **positive reps**; ...
Properties of (symplectic) Anosov representations

Theorem (Labourie, Guichard-Wienhard)

Given \( \rho : \pi_1(\Sigma) \longrightarrow \text{Sp}(2n, \mathbb{K}) \) \( Q_1 \)-Anosov, then

- \( \rho \) is discrete and faithful.
- \( \rho \) acts proximally on \( \mathbb{P}(\mathbb{K}^{2n}) \) (that is, \( \forall \gamma \in \pi_1(\Sigma), \exists x^+, x^- \in \mathbb{P}(\mathbb{K}^{2n}) \) s.t. \( \forall y \in \mathbb{P}(\mathbb{K}^{2n}) \) transverse to \( x^\gamma \rho(\gamma^{\pm n})y \longrightarrow x^\gamma \)).
- the orbit map \( \pi_1(\Sigma) \longrightarrow \text{Sp}(2n, \mathbb{K})/K \) is a quasi-isometric embedding wrt the word distance and the Riemannian distance, resp.

In addition, \( \mathcal{X}_{Q_1}(\pi_1(\Sigma), \text{Sp}(2n, \mathbb{K})) \) is open and \( \text{Mod}(\Sigma) \) acts prop. disc. on \( \mathcal{X}_{Q_1}(\pi_1(\Sigma), \text{Sp}(2n, \mathbb{K})) \).
Domain of discontinuity

Given $\rho$ Q1–Anosov, we have $\xi_1^\rho: \partial_\infty(\pi_1(\Sigma)) \rightarrow \mathbb{CP}^{2n-1}$.

$\forall \ell \in \mathbb{CP}^{2n-1}$ let $K_\ell = \{W \in \text{Lag}(\mathbb{C}^2^n) | \ell \subset W\} \subset \text{Lag}(\mathbb{C}^{2n})$.

Define

$$K_{\xi_1^\rho} = \bigcup_{t \in \partial_\infty(\pi_1(\Sigma))} K_{\xi_1^\rho}(t) \quad \text{and} \quad \Omega_{\xi_1^\rho} = \text{Lag}(\mathbb{C}^{2n}) \setminus K_{\xi_1^\rho}$$

**Theorem (Guichard–Wienhard)**

$\Omega_{\xi_1^\rho}$ is a domain of discontinuity for the action of $\rho$ on $\text{Lag}(\mathbb{C}^{2n})$ (that is, $\Omega_{\xi_1^\rho}$ is open and $\rho$ acts on it freely, properly discontinuously and cocompactly).

**Question**

What is the topology of $\Omega_{\xi_1^\rho}/\rho$?
Main theorem

**Conjecture (Dumas-Sanders)**

Given $\rho: \pi_1(\Sigma) \rightarrow G$ B–quasi-Hitchin, then $\Omega_{\xi_\rho}/\rho$ is a fiber bundle over a surface with fiber a compact Poincaré duality space.

Dumas - Sanders prove the conjecture for $G = \text{SL}(3, \mathbb{C})$.

**Theorem (Alessandrini - M. - Wienhard)**

Given $\rho: \pi_1(\Sigma) \rightarrow \text{Sp}(4, \mathbb{C})$ $Q_1$–quasi-Hitchin, then $\Omega_{\xi_1\rho}/\rho$ is a fiber bundle over a surface with fiber $F$ and structure group $\text{SO}(2)$ and Euler class $2g - 2$. In addition, the fiber $F$ is homeomorphic to a quotient of $(\mathbb{S}^2 \times \mathbb{S}^2)/A_4$.

The cont. projection $\Omega_{\xi_1\rho} \rightarrow \mathbb{H}^2$ (which descends to $\Omega_{\xi_1\rho}/\rho \rightarrow \Sigma$) comes from the study of the space $\text{Lag}(\mathbb{C}^4)$ and its $\text{SL}(2, \mathbb{C})$–orbits.
**SL(2, C)–orbits**

First, we study the action of \( SL(2, \mathbb{C}) \) on \( \text{Lag} (\mathbb{C}^4) \).

Recall that \( \mathbb{C}^4 = \mathbb{C}^{(3)}[X, Y] \) and \( SL(2, \mathbb{C}) \) acts on \( \mathbb{C}^4 \) by acting on the roots of the polynomials.

Consider the Veronese embeddings:

- \( \xi^1 : \mathbb{R}P^1 \longrightarrow \mathbb{C}P^3 \) 
  
  \[ [a : b] \mapsto (bX - aY)^3 \]

  which can be extended to

- \( \xi^1_\mathbb{C} : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^3 \);

- \( \xi^2 : \mathbb{R}P^1 \longrightarrow \text{Lag} (\mathbb{C}^4) \)

  \[ [a : b] \mapsto \langle (bX - aY)^3, (bX - aY)^2 X \rangle. \]

  \( \xi^2_\mathbb{C} : \mathbb{C}P^1 \longrightarrow \text{Lag}(\mathbb{C}^4). \)

Recall that \( \forall \ell \in \mathbb{C}P^3, K_\ell = \{ W \in \text{Lag}(\mathbb{C}^4) | \ell \subset W \} \subset \text{Lag}(\mathbb{C}^4) \).

**Question**

*What are the SL(2, C)–orbits in \( \text{Lag}(\mathbb{C}^4) \)?
**SL(2, ℂ)–orbits of Lag(ℂ⁴)**

**Proposition**

*There are 3 SL(2, ℂ)–orbits in Lag(ℂ⁴):*

- Lag(ℂ⁴) \ K_{ξ₁^C} \cong ℳ = \{ideal regular hyp. tetrahedra\} (open orbit).
- K_{ξ₁^C} \setminus ξ₂^C(ℂP¹);
- ξ₂^C(ℂP¹) (closed orbit).

Recall that an ideal hyperbolic tetrahedra is regular ⇐⇒ it has max volume ⇐⇒ the cross-ratio of its vertices is \(\frac{1-i\sqrt{3}}{2}\).

Note that \(K_{ξ₁^C}\) corresponds to “degenerate” ideal regular tetrahedra.
### Sketch of the proof

\[ K_{\xi_1} \cong \mathbb{C}P^1 \times \mathbb{C}P^1. \]

\[ K_{\xi_1} = \bigcup_{t \in \mathbb{C}P^1} K_{\xi_1}(t) = \{ W \in \text{Lag}(\mathbb{C}^4) \mid \exists p = (X - z_0 Y)^3 \in W \} = \{ W \in \text{Lag}(\mathbb{C}^4) \mid \forall p \in W, p(X, Y) = (X - z_0 Y)q(X, Y) \}. \]

So \( F : \mathbb{C}P^1 \times \mathbb{C}P^1 \xrightarrow{\cong} K_{\xi_1} \) by \( F([a : b], [c : d]) = \)

\[ \begin{cases} 
< (bX - aY)^3, (bX - aY)^2X > = \xi_2^2([a : b]) & \text{if } [a : b] = [c : d] \\
< (bX - aY)^3, (dX - cY)^2(bX - aY) > & \text{if } [a : b] \neq [c : d] 
\end{cases} \]

### Remark

\[ K_{\xi_1} \cong \mathbb{R}P^1 \times \mathbb{C}P^1. \]
Sketch of the proof (summary)

\[ \text{Lag}(\mathbb{C}^4) \setminus K_{\xi_C} \cong \mathcal{Z}. \]

- \( \forall W \in \text{Lag}(\mathbb{C}^4) \setminus K_{\xi_C}, \exists p \in W \) with a double root. Up to \( \text{SL}(2, \mathbb{C}) \), we can suppose \( p = X^2 Y \).
- We study all the Lagrangians containing \( p = X^2 Y \).
- \( \forall W \in \text{Lag}(\mathbb{C}^4) \setminus K_{\xi_C} \), \( \exists 4 p \in W \) with double roots and these 4 roots form a regular ideal hyperbolic tetrahedra.
Sketch of the proof (details)

\[ \text{Lag}(\mathbb{C}^4) \setminus K_{\xi_1} \cong \mathcal{I}. \]

- \( \forall W \in \text{Lag}(\mathbb{C}^4), \exists p \in W \text{ s.t.} \)
  \[ p(X, Y) = (X - z_0 Y)^2(X - z_1 Y), \text{ with } z_i \in \mathbb{CP}^1. \]
  - Let \( z_0 = 0 \) and \( z_1 = \infty \), so \( p = X^2 Y. \)
  \[ L_{X^2 Y} = \{ W \in \text{Lag}(\mathbb{C}^4) \mid X^2 Y \in W \} \]
  \[ = \{ W = \langle X^2 Y, X^3 + \frac{b}{a} Y^3 \rangle \mid \frac{b}{a} \in \mathbb{CP}^1 \}. \]

By using the action of \( \text{SL}(2, \mathbb{C}) \), we can assume \( \frac{b}{a} = 1 \) and study \( W = \langle X^2 Y, X^3 + Y^3 \rangle. \) So

\[ K_{\xi_1} \mathbb{C} = \text{SL}(2, \mathbb{C}) \cdot \langle X^2 Y, X^3 + Y^3 \rangle. \]
Sketch of the proof (details)

\[
\text{Lag}(\mathbb{C}^4) \setminus K_{\xi_1} \cong \mathbb{C}.
\]

- In \( W = \langle X^2 Y, X^3 + Y^3 \rangle \) \( \exists 4 \) ‘special’ polynomials with double roots. Their associate double and single roots are:
  
  (i) \( z_0 = 0 \) and \( w_0 = \infty \);
  
  (ii) \( z_1 = \sqrt[3]{2} \) and \( w_1 = -\frac{1}{3^{\frac{3}{4}}} \);
  
  (iii) \( z_2 = \frac{-1-i\sqrt{3}}{3^{\frac{3}{4}}} \) and \( w_2 = \frac{1+i\sqrt{3}}{2^{\frac{3}{4}}} \);
  
  (iv) \( z_3 = \frac{-1+i\sqrt{3}}{3^{\frac{3}{4}}} \) and \( w_3 = \frac{1-i\sqrt{3}}{2^{\frac{3}{4}}} \).

(Proof: use the notion of discriminant.)

- The cross ratio is \( [z_0 : z_1 : z_2 : z_3] = [w_0 : w_1 : w_2 : w_3] = \frac{1-i\sqrt{3}}{2} \).
**Main theorem**

By sending a tetrahedra into its barycenter (or its degenerations), we define the cont. projections

\[ \text{Lag}(\mathbb{C}^4) \rightarrow \mathbb{H}^3 \cup \mathbb{CP}^1 \quad \text{and} \quad \Omega_{\xi_1} \rightarrow \mathbb{H}^2, \]

where

\[ \Omega_{\xi_1} = \mathcal{T} \cup (\mathbb{CP}^1 \setminus \mathbb{RP}^1) \times 
\mathbb{RP}^1 \rightarrow \mathbb{H}^3 \cup (\mathbb{CP}^1 \setminus \mathbb{RP}^1) \rightarrow \mathbb{H}^2 \]

**Theorem**

Given \( \rho : \pi_1(\Sigma) \rightarrow \text{Sp}(4, \mathbb{C}) \) \( Q_1 \)-Anosov, then \( \Omega_{\xi_1} / \rho \) is a fiber bundle over a surface with fiber \( F \) and structure group \( \text{SO}(2) \) and Euler class \( 2g - 2 \).

**Question**

*What can we say about the fiber \( F \) ?*
**What can we say about $F$?**

**Theorem**

*The fiber $F$ is homeomorphic to a quotient of $(S^2 \times S^2)/A_4$.*

Let’s describe first $S^2 \times S^2$ via mapping cylinders:

- Let $M_p = T^1(S^2) \times [0, 1]/(T^1(S^2) \times \{0\} \sim S^2)$ via the projection $p: T^1(S^2) \longrightarrow S^2$.
- Then $M_p \sqcup_{id} M_p/ \sim \cong S^2 \times S^2$.

$T^1(S^2)/A_4 \cong T^{1,orb}(S^2(2, 3, 3))$ (reg. tetrahedra with fixed barycenter).

If we do the same construction replacing $T^1(S^2)$ with $T^{1,orb}(S^2(2, 3, 3))$ and $p$ with a map $\hat{p}: T^{1,orb}(S^2(2, 3, 3)) \longrightarrow S^2(2, 3, 3)$ we obtain $X = (S^2 \times S^2)/A_4$. The fiber $F$ is a quotient of $X$ and is homotopically equivalent to it.
**Theorem (Wolf J.)**

*There are 6 $\text{Sp}(4, \mathbb{R})$–orbits in $\text{Lag}(\mathbb{C}^4)$:*

$$\mathcal{R}_i = \{ \mathcal{W} \in \text{Lag}(\mathbb{C}^4) \mid \dim(\mathcal{W} \cap \overline{\mathcal{W}}) = i \}.$$ 

*Then*

- $\mathcal{R}_0 = \mathcal{H}_{2,0} \cup \mathcal{H}_{1,1} \cup \mathcal{H}_{0,2}$ where $\mathcal{H}_{i,j} \cong X_{i,j} = \text{Sp}(4, \mathbb{R})/U(i,j)$ (open).
- $\mathcal{R}_1$ fibers over $\mathbb{P}(\mathbb{R}^4)$ with fiber isomorphic to $X_{0,1} \cup X_{1,0}$.
- $\mathcal{R}_2 \cong \text{Lag}(\mathbb{R}^4)$ (closed).
Sketch of the proof

Proof.

1. **$R_0$:**
   - $\omega_\mathbb{C}$ defines a non-degenerate $Sp(4, \mathbb{R})$–invariant Hermitian form $h$:
     \[ h(v, w) := i\omega_\mathbb{C}(\bar{v}, w); \]
   - $R_0 = \mathcal{H}_{2,0} \cup \mathcal{H}_{1,1} \cup \mathcal{H}_{0,2}$ w/ \[ \mathcal{H}_{p,q} = \{ W \in R_0 \mid h\big|_{W \times W} \text{ has signature } (p, q) \}. \]

2. **$R_1$:**
   - $\forall W \in R_1$, then $Z = W \cap \overline{W}$ is the complexification of $Z' \in \mathbb{P}(\mathbb{R}^4)$ and this gives $p : R_1 \rightarrow \mathbb{P}(\mathbb{R}^4)$;
   - Let $M = Z^{\perp_{\omega_\mathbb{C}}}/Z$ is a 2–dim. sympl. space. Any $W \in p^{-1}(Z')$ is uniquely determined by $Y \in \text{Lag}(M)$ s.t. $Y \cap \overline{Y} = \{0\}$.

3. **$R_2$:** any $W \in R_2$ is the complexification of $W' \in \text{Lag}(\mathbb{R}^4)$.
Ralationship with $\Omega_{\xi^1}$

Question

*What is the relationship between $F$ and the $\text{Sp}(4, \mathbb{R})$–orbits?*
Open questions

- What can we say about the d.o.d. $\Omega \subset \mathcal{F}(\mathbb{C}^4)$ for a Fuchsian representation $\rho: \pi_1(\Sigma) \to \text{SL}(2, \mathbb{R}) \to \text{Sp}(4, \mathbb{C})$?

- What is the connection with Dumas-Sanders’ work? What can we say for $G = \text{Sp}(2n, \mathbb{C})$ or other cases? (We are working on this with D. Dumas.)

- What can we say about the quotient of the symmetric space?

- What can we say about limit of these representations? Can you combine punctured Fuchsian groups in order to understand ‘geometrically finite’ groups? Are there ‘geometrically infinite’ groups?

- Can we use other methods to find a fibration? (We plan to work on this with Q. Li using Higgs bundles techniques.)
End
Cartan decomposition and contraction properties

Let $\mathfrak{a} = \{ \text{diag}(\lambda_1, \ldots, \lambda_n, -\lambda_n, \ldots, -\lambda_1) \mid \lambda_i \in \mathbb{R} \} \subset \mathfrak{sp}$. Decompose $\text{Sp}(2n, \mathbb{K}) = K \exp(\mathfrak{a}) K$. [Problem: not unique!]

Given $\mathfrak{a}^+ = \{ \text{diag}(\lambda_1, \ldots, \lambda_n, -\lambda_n, \ldots, -\lambda_1) \mid \lambda_1 \geq \lambda_2 \ldots \lambda_n \geq 0 \}$, then $\text{Sp}(2n, \mathbb{K}) = K \exp(\mathfrak{a}^+) K$ is unique:

$\forall g \in \text{Sp}(2n, \mathbb{K}), \exists! \; a_g \in \mathfrak{a}^+ \text{ s.t. } k_1 \exp(a_g) k_2$, where

**Definition**

$\mu : \text{Sp}(2n, \mathbb{K}) \longrightarrow \mathfrak{a}^+$ defined by $g \mapsto a_g$ is called the *Cartan projection* of $\text{Sp}(2n, \mathbb{K})$.

Let $\alpha_i := \epsilon_i - \epsilon_{i+1} \in \mathfrak{a}^*$ and $\alpha_n = 2\epsilon_n \in \mathfrak{a}^*$, where $\epsilon_i (\text{diag}(\lambda_1, \ldots, \lambda_n, -\lambda_n, \ldots, -\lambda_1)) = \lambda_i$.

Then contraction properties is: $\forall \gamma_n \in \pi_1(\Sigma), \lim_{n \to \infty} \alpha_i (\mu(\rho(\gamma_n))) = \infty$. 
Sketch of the proof (cont.)

\[ \text{Lag}(\mathbb{C}^4) \setminus K_{\xi_1} \cong \mathbb{S}. \]

- \( \forall W \in \text{Lag}(\mathbb{C}^4) \setminus K_{\xi_1}, \exists p \in W \) with a double root. Up to \( \text{SL}(2, \mathbb{C}) \), we can suppose \( p = X^2Y \).
- We study all the Lagrangians containing \( p = X^2Y \).
- \( \forall W \in \text{Lag}(\mathbb{C}^4) \setminus K_{\xi_1} \exists 4 \ p \in W \) with double roots and these 4 roots form a regular ideal hyperbolic tetrahedra.
Sp(2n, ℝ)–orbits

Let $\text{Is}_i(\mathbb{R}^{2n}) = \{i - \text{dimensional isotropic subspaces}\}$.

**Theorem (Wolf J.)**

$Lag(\mathbb{C}^{2n}) = \bigcup_{i=0}^{n} \mathcal{R}_i$, where

$$\mathcal{R}_i = \{W \in Lag(\mathbb{C}^{2n}) \mid \dim(W \cap \overline{W}) = i\}.$$

Then

- $\mathcal{R}_0 = \bigcup_{p=0}^{n} \mathcal{H}_{p,n-p}$, where $\mathcal{H}_{p,n-p} \cong X_{p,n-p}$ (open).
- $\mathcal{R}_i$ fibers over $\text{Is}_i(\mathbb{R}^{2n})$ with fiber isomorphic to $\bigcup_{p=0}^{n-i} X_{p,n-i-p}$.
- $\mathcal{R}_n \cong \text{Lag}(\mathbb{R}^{2n})$ (closed).
Sketch of the proof

Proof.

1. $\mathcal{R}_0$:
   - $\omega_C$ defines a non-degenerate $Sp(2n, \mathbb{R})$–invariant Hermitian form $h$:
     \[
     h(v, w) := i\omega_C(\overline{v}, w) \quad \forall v, w \in \mathbb{C}^{2n};
     \]
   - $\mathcal{R}_0 = \bigcup_{p=0}^{n} \mathcal{H}_{p,n-p}$, where
     \[
     \mathcal{H}_{p,q} = \{ W \in \mathcal{R}_0 \mid h|_{W \times W} \text{ has signature } (p, q) \}.
     \]

2. $\mathcal{R}_i$:
   - $\forall W \in \mathcal{R}_i$, then $Z = W \cap \overline{W}$ is the complexification of $Z' \in Is_i(\mathbb{R}^4)$ and this gives $p: \mathcal{R}_i \longrightarrow Is_i(\mathbb{R}^4)$;
   - Let $M = Z^\perp \omega_C / Z$. Any $W \in p^{-1}(Z')$ is uniquely determined by $Y \in \text{Lag}(M)$ s.t. $Y \cap \overline{Y} = \{0\}$.

3. $\mathcal{R}_n$: any $W \in \mathcal{R}_n$ is the complexification of $W' \in \text{Lag}(\mathbb{R}^{2n})$.