Anosov representations

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If S_1 and S_2 are finite generating sets of Γ, then $\mathrm{id} : (\Gamma, d_{S_1,\Gamma}) \to (\Gamma, d_{S_2,\Gamma})$ is a quasi-isometric embedding.

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4. Γ *is hyperbolic, and the bundle* $E := U\Gamma \times_\rho \partial M^2$ *over* $U\Gamma$ *has "suitable contracting and expanding properties".*

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- Schottky groups $F_n \subset G$.

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Theorem (Gromov)

Suppose that $\text{rank}(G) = 1$*. For any* $L, A > 0$*, there exists* $K, L', A' > 0$ *so that every* (L, A, K)*-local quasi-geodesic in* M *is a* (L', A')*-quasi-geodesic.*

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$\Rightarrow$ The orbit map being a quasi-isometric embedding is stable under perturbation when $\text{rank}(G) = 1$.

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There is a basis of linear functionals $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subset \mathfrak{a}^*$ so that

- W is generated by reflections along $L_1, \ldots, L_n$, where
  $L_i := \exp_M(\ker(\alpha_i))$ for all i.

Properties of M when $\mathrm{rank}(G) \geq 2$.

- G acts transitively on the space of maximal flats in M.
- For any maximal flat $F \subset M$, $\mathrm{Stab}_G(F)$ acts transitively on F.
- G does not act transitively on the space of geodesics.
  $\Rightarrow$ distance is not a good invariant of a pair of points in M.

Recall that $F_0 := \exp_M(\mathfrak{a}) \subset M$ is a maximal flat containing p_0.

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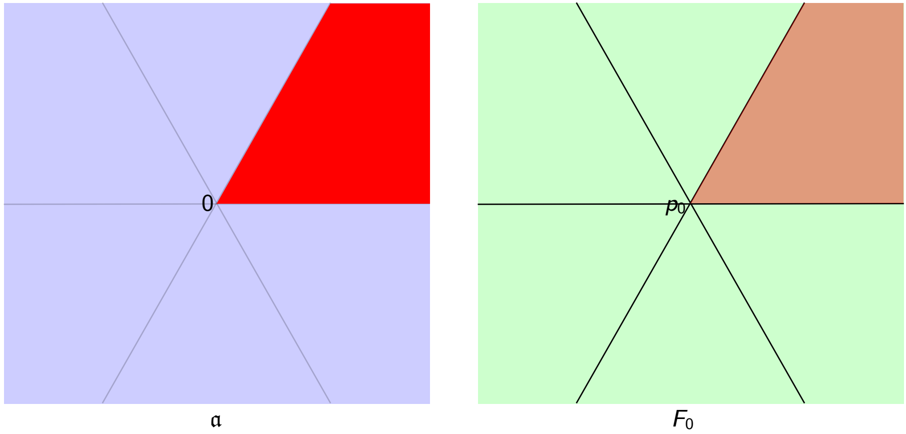
- W is generated by reflections along $L_1, \ldots, L_n$, where $L_i := \exp_M(\ker(\alpha_i))$ for all i.
- $\exp_M(\overline{\mathfrak{a}^+})$ is a fundamental domain of the W action on F_0, where

$$\overline{\mathfrak{a}^+} := \{X \in \mathfrak{a} : \alpha(X) \geq 0 \text{ for all } \alpha \in \Delta\}.$$

Weyl group action on F_0 when $G = \mathrm{SL}(3, \mathbb{R})$



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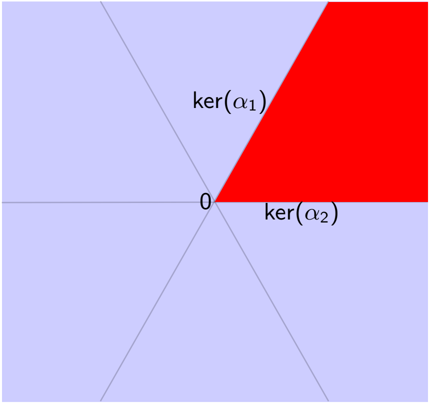

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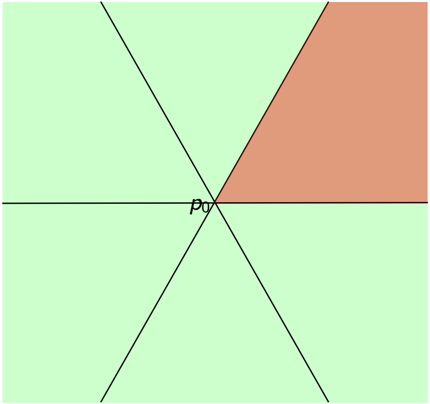

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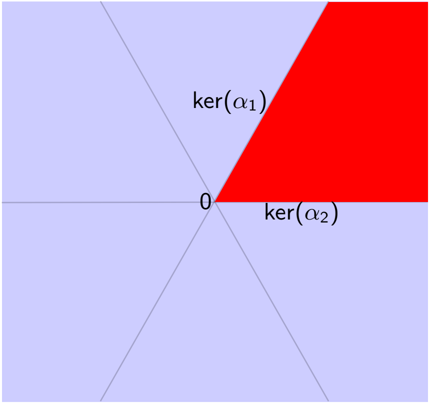

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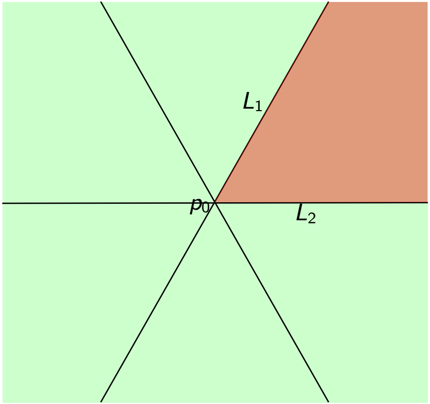

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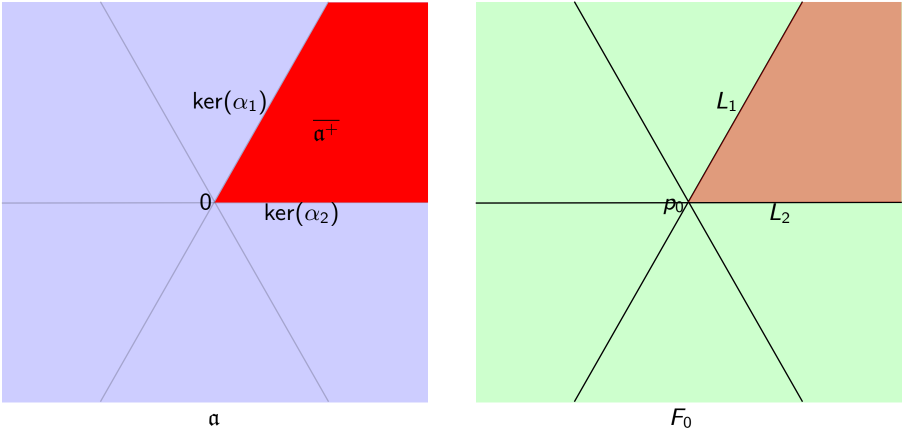

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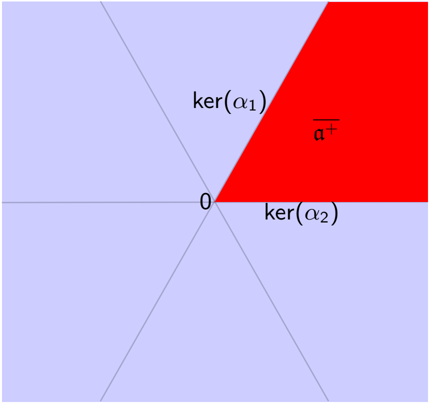

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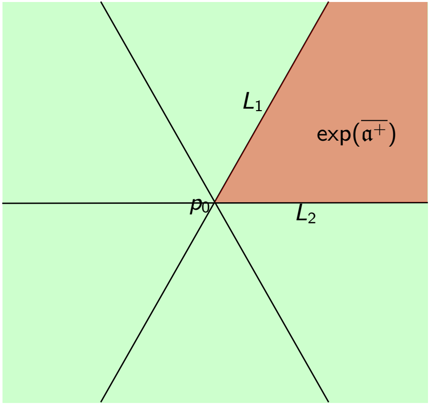

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Uniform regularity

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Upshot: Given any $p, q \in M$, there is a unique $A(p,q) \in \overline{\mathfrak{a}^+}$ so that $g \cdot p = p_0$ and $g \cdot q = \exp_M(A(p,q))$ for some $g \in G$.

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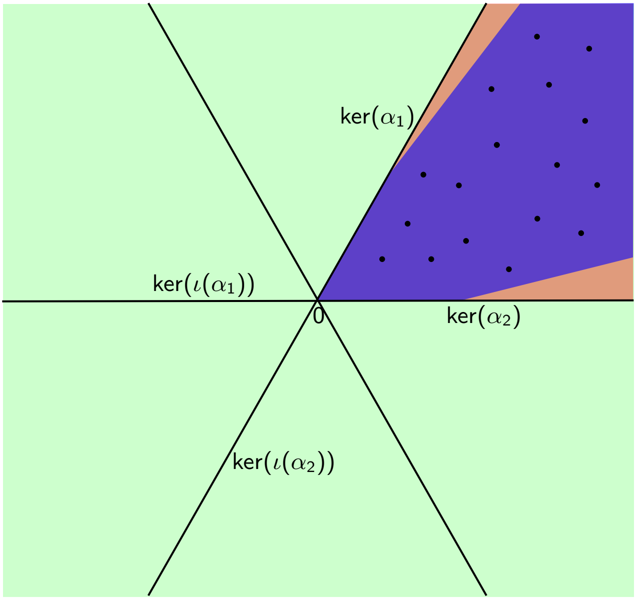
A representation $\rho : \Gamma \to G$ is θ-Anosov if ρ has finite kernel and $\rho(\Gamma) \subset G$ is θ-URU.

URU subgroup



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For all L, A, C, c, C', c', S, there exists K, L', A', S' so that every (c, C, S, θ)-regular (L, A, K)-local quasi-geodesic is a (c', C', S', θ)-regular (L', A')-quasi-geodesic.

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$\Rightarrow$ θ-Anosovness is stable under perturbation.

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Theorem (Burger-Iozzi-Labourie-Wienhard)

Any maximal representation is $\{\alpha\}$-Anosov for some $\alpha \in \Delta$.

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Properties of these representations were studied by Barbot and Mérigot.

The End