Basmajian's identity in higher Teichmüller theory

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Theorem (Basmajian)

$$\ell(\partial X) = \sum_{\alpha \in \mathcal{O}(X)} 2 \log \left(\coth \left(\frac{\ell(\alpha)}{2} \right) \right)$$

Today's goal

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Theorem (V. – Yarmola)

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Basmajian's identity extends to Hitchin representations.

Need to find correct analogues for length, orthogeodesics, and the summands in the original identity.

Hitchin representations

Let Σ be a surface possibly with boundary.

Definition

- An *n-Fuchsian homomorphism* is a composition of homomorphisms $\pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbb{R}) \to \mathrm{PSL}(n, \mathbb{R})$, where the first homomorphism is Fuchsian and the second is the unique irreducible representation $\mathrm{PSL}(2, \mathbb{R}) \to \mathrm{PSL}(n, \mathbb{R})$.

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- An *n-Hitchin homomorphism* is a homomorphism $\pi_1(\Sigma) \to \mathrm{PSL}(n,\mathbb{R})$ that can be continuously deformed to an n-Fuchsian homomorphism while keeping the image of each boundary component purely loxodromic at each stage of the deformation.

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- The length of an element $\gamma \in \pi_1(\Sigma)$ with respect to a Hitchin representation ρ is

$$\ell_\rho(\gamma) = \log \left| \frac{\lambda_{max}(\rho(\gamma))}{\lambda_{min}(\rho(\gamma))} \right|,$$

where λ_{max} (λ_{min}) is the eigenvalue with the largest (smallest) absolute value.

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  Note: For $n = 2$ or $n = 3$, this agrees with the translation length in the associated hyperbolic metric or Hilbert metric, respectively.

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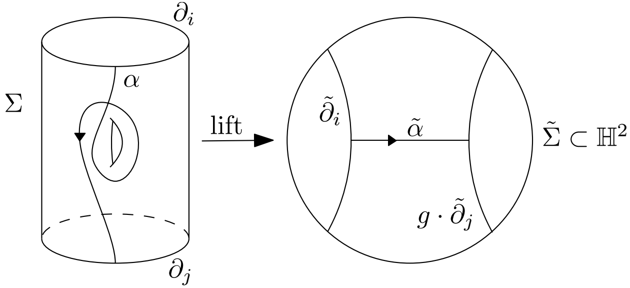

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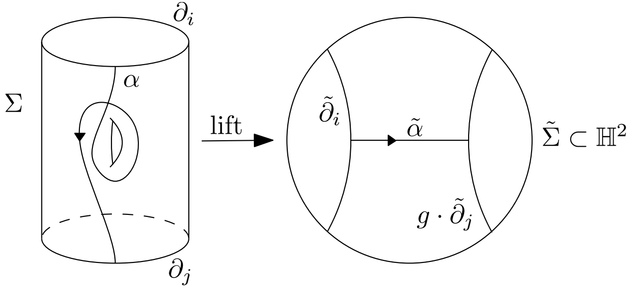

Observe that replacing g with an element of the form $\partial_i^k \cdot g \cdot \partial_j^l$ also yields α.

Orthospectrum

Proposition

Let

$$\mathcal{O}(\Sigma, \mathcal{A}) = \{\langle \partial_i \rangle g \langle \partial_j \rangle : i, j \in \{1, \ldots, m\}, g \in \pi_1(\Sigma), \langle \partial_i \rangle g \langle \partial_j \rangle \neq \langle \partial_i \rangle\},$$

then $\{\text{orthogeodesics}\} \longleftrightarrow \mathcal{O}(\Sigma, \mathcal{A})$.

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then $\{\text{orthogeodesics}\} \longleftrightarrow \mathcal{O}(\Sigma, \mathcal{A})$.

Note: If Σ has a single boundary component, then $\mathcal{O}(\Sigma, \{\partial\})$ can be identified with the nontrivial elements of

$$\pi_1(\partial\Sigma)\backslash\pi_1(\Sigma)/\pi_1(\partial\Sigma).$$

Summands

Fix $\langle \partial_i \rangle g \langle \partial_j \rangle$ and let $\rho : \pi_1(\Sigma) \to \mathrm{PSL}(n, \mathbb{R})$ be a Hitchin representation.

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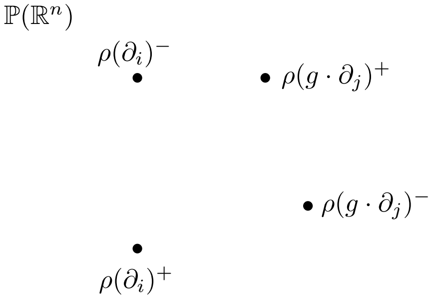

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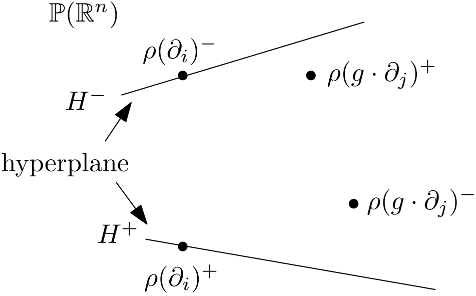

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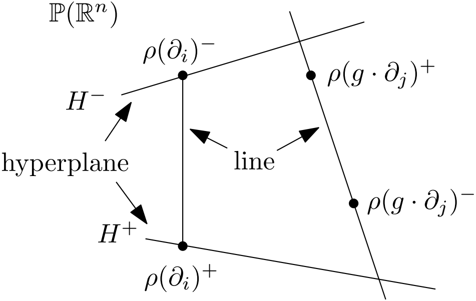

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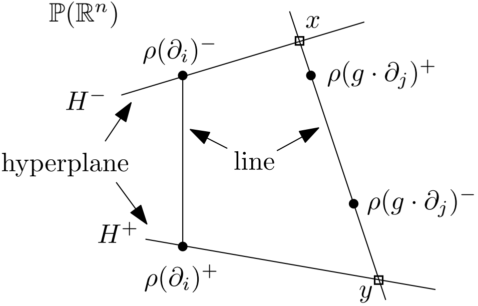

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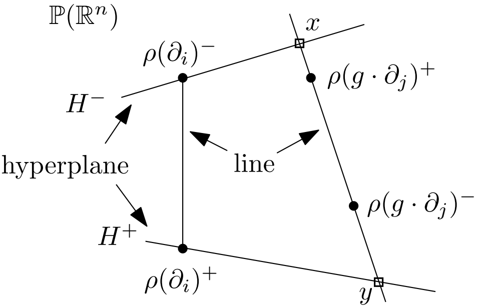

Define

$$F_\rho(\langle\partial_i\rangle g\langle\partial_j\rangle) = \log b(y, \rho(g\cdot\partial_j)^+, x, \rho(g\cdot\partial_j)^-),$$

where b is the standard cross ratio on a line.

Basmajian's identity

Theorem (V. – Yarmola)

Let Σ be a compact orientable surface with nonempty boundary whose double has genus at least 2. If ρ is a Hitchin representation of $\pi_1(\Sigma)$, then

$$\ell_\rho(\partial\Sigma) = \sum_{\langle\partial_i\rangle g\langle\partial_j\rangle \in \mathcal{O}(\Sigma,\mathcal{A})} F_\rho(\langle\partial_i\rangle g\langle\partial_j\rangle).$$

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- If $n = 2$, then this identity is Basmajian's identity.
- If $n = 3$, then this identity has an interpretation in terms of convex real projective structures, the Hilbert metric, and orthogonal projections.

Basmajian's proof

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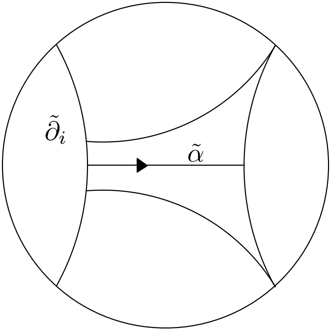

$\tilde{X} \subset \mathbb{H}^2$

Lebesgue measure

Theorem (Labourie)

Let S *be a closed surface and* $\rho : \pi_1(S) \to \mathrm{PSL}(n, \mathbb{R})$ *a Hitchin representation, then there exists a* ρ*-equivariant limit map* $\xi_\rho : \partial\pi_1(S) \to \mathbb{P}(\mathbb{R}^n)$*. Further, the image of* ξ_ρ *is* C^1*.*

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Let S be a closed surface and $\rho : \pi_1(S) \to \mathrm{PSL}(n, \mathbb{R})$ a Hitchin representation, then there exists a ρ-equivariant limit map $\xi_\rho : \partial\pi_1(S) \to \mathbb{P}(\mathbb{R}^n)$. Further, the image of ξ_ρ is C^1.

Theorem (V. – Yarmola)

Let S be an oriented closed surface and $\Sigma \subset S$ an incompressible subsurface. Let ρ be a Hitchin representation of S and ξ_ρ the associated limit curve. If μ_ρ is the pullback of the Lebesgue measure on the image of ξ_ρ, then $\mu_\rho(\partial\pi_1(\Sigma)) = 0$.

Thanks

Thanks for listening!