Positively ratioed representations and length functions in higher rank

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Geodesic currents

Let S be a closed orientable surface of genus g at least 2 with fundamental group $\Gamma := \pi_1(S)$.
Let $\mathcal{G}(S)$ be the set of unoriented geodesics in $\widetilde{S}$.

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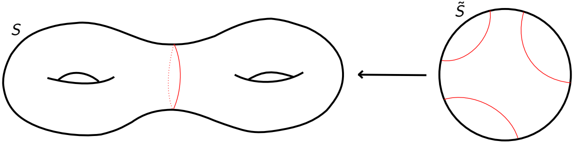
Example: Dirac measure on the lifts of a (primitive) closed geodesic on S.



Rank 1 case

Theorem (Bonahon)

1. *There exists a continuous, symmetric, bilinear function*

$$i\colon \mathcal{C}(S) \times \mathcal{C}(S) \to \mathbb{R}^+$$

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2. $\mathcal{T}(S)$ *embeds in* $\mathcal{C}(S)$*. If* $\rho\colon \Gamma\to \mathrm{PSL}(2,\mathbb{R})$ *is in* $\mathcal{T}(S)$ *with image* L_ρ *and* $c\in\mathcal{CG}(S)$,

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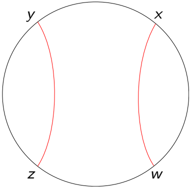
3. $\mathcal{T}(S)$ *embeds in* $\mathcal{PC}(S)$, *which is compact.*

$$\overline{\mathcal{T}(S)}^{\mathcal{PC}(S)} = \textit{Thurston compactification.}$$

Recipe for geodesic currents



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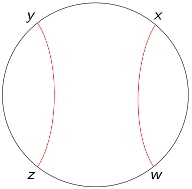

$$L_\rho([x,y]\times[z,w]) = \log b_\rho(x,y,z,w) = \log\left(\frac{x-z}{x-w}\frac{y-w}{y-z}\right)$$

We can write the length of $c \in \mathcal{CG}(S)$ as

$$i(L_\rho, c) = L_\rho([\gamma_c \cdot a, a]\times[\gamma_c^-, \gamma_c^+]) = \ell^\rho_{\mathrm{hyp}}(c) = \log\frac{\lambda_1^\rho(\gamma_c)}{\lambda_2^\rho(\gamma_c)}.$$

Main question

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Can we understand length functions in higher rank by generalizing this picture?

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1. What kind of representations do we want to consider?
2. What kind of length functions do we want to consider?
3. How does one associates to this data a geodesic current?

An example: $G = \mathrm{PSL}(n, \mathbb{R})$

Theorem (Labourie - Guichard)

$\rho\colon \Gamma \to G$ is a Hitchin representation if and only if there exist continuous, ρ-equivariant maps

$$\xi\colon \partial\Gamma \to \mathrm{Flag}(\mathbb{R}^n)$$
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Remark: $\mathrm{Flag}(\mathbb{R}^n) = G/P_\Delta$ where

$$P_\Delta = \{\text{upper triangular matrices}\}.$$

Length functions

Theorem (Labourie)

For every $c \in \mathcal{CG}(S)$, for any representative $\gamma_c \in \Gamma$ there is a lift of $\rho(\gamma_c)$ to $\mathrm{SL}(n, \mathbb{R})$ with distinct positive eigenvalues

$$\lambda_1^\rho(c) > \lambda_2^\rho(c) > \dots \lambda_n^\rho(c) > 0.$$

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Problem (?)

Since we are considering unoriented geodesics, our length functions need to satisfy $\ell(\gamma_c) = \ell(\gamma_c^{-1})$.

Length functions

Definition

The *k-length* of $c \in \mathcal{CG}(S)$ is

$$\ell_k^\rho(c) = \log\left(\frac{\lambda_1^\rho(c)\dots\lambda_k^\rho(c)}{\lambda_{n-k+1}^\rho(c)\dots\lambda_n^\rho(c)}\right)$$

for $k = 1, 2, \dots, n-1$.

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In other (more theoretical) words

$$\ell_k^\rho(c) = (\omega_{\alpha_k} + \omega_{\alpha_{n-k}}) \circ \lambda \circ \rho(\gamma_c)$$

where ω_{α_k} is the *k*-th restricted fundamental weight and λ is the Lyapunov projection.

Part II

Geodesic currents for Hitchin representations

Cross ratios

Definition (Ledrappier)

A Hölder cross ratio is a Hölder continuous function $B\colon \partial\Gamma^{[4]} \to \mathbb{R}$, invariant under the action of Γ and such that

a. $B(x, y, z, w) = B(z, w, x, y)$,

b. $B(x, y, z, w) = B(x, y, z, u) + B(x, y, u, w)$.

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Lemma (Hämenstadt, Otal, Quint, Sambarino, M.-Zhang)

For every $k \in \{1,2,\ldots,n-1\}$, there is a unique cross ratio B_k^ρ with

$$B_k^\rho(\gamma_c \cdot a, a, \gamma_c^-, \gamma_c^+) = \ell_k^\rho(c)$$

for every $c \in \mathcal{CG}(S)$, $a \in \partial\Gamma \setminus \{\gamma^-, \gamma^+\}$.

The explicit formula for B_k^ρ

$$B_k^\rho(x,y,z,w) := \log\left|\frac{\xi(x)^{(n-k)}\wedge\xi(z)^{(k)}}{\xi(x)^{(n-k)}\wedge\xi(w)^{(k)}}\frac{\xi(y)^{(n-k)}\wedge\xi(w)^{(k)}}{\xi(y)^{(n-k)}\wedge\xi(z)^{(k)}}\right| + \log\left|\frac{\xi(z)^{(n-k)}\wedge\xi(x)^{(k)}}{\xi(z)^{(n-k)}\wedge\xi(y)^{(k)}}\frac{\xi(w)^{(n-k)}\wedge\xi(y)^{(k)}}{\xi(w)^{(n-k)}\wedge\xi(x)^{(k)}}\right|.$$

A missing piece - Positively ratioed representations

Lemma

For any four points $x, y, z, w \in \partial\Gamma$ *in this cyclic order along* $\partial\Gamma$ *the cross ratio* $B_k^\rho(x, y, z, w) \geq 0$.

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Theorem (Hämenstadt, M.-Zhang)

For any Hitchin representation ρ *and for any* $k = 1, 2, \ldots, n-1$, *there exists a unique geodesic current* ν_k^ρ *such that*

$$i(\nu_k^\rho, c) = B_k^\rho(\gamma_c \cdot a, a, \gamma_c^-, \gamma_c^+) = \ell_k^\rho(c)$$

and $c \in \mathcal{CG}(S)$.

Part IV

First results

Asymptotic behavior of lengths

Let $\{\rho_j : \Gamma \to \mathrm{PSL}(n_j, \mathbb{R})\}_{j=1}^{\infty}$ be a sequence of Hitchin representations, and let $k_j \in \{1, \ldots, n_j - 1\}$.

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Proposition

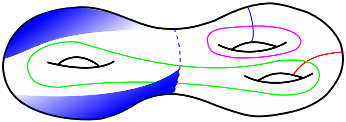
Up to passing to a subsequence, there is

- a subsurface $S' \subset S$,
- pairwise non-intersecting, non-peripheral simple closed curves $\{c_1, \ldots, c_r\}$ in $\mathcal{CG}(S \setminus S')$

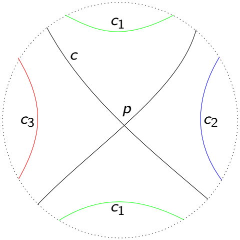
so that $A := S' \cup \bigcup_{s=1}^{r} c_s$ is non-empty, and the following holds.

If c intersects A non-trivially

1. $\lim_{j\to\infty} \frac{\ell_{k_j}^{\rho_j}(d)}{\ell_{k_j}^{\rho_j}(c)} = 0$ if d does not intersect A.
2. $\lim_{j\to\infty} \frac{\ell_{k_j}^{\rho_j}(d)}{\ell_{k_j}^{\rho_j}(c)} \in \mathbb{R}^+$ if d intersect A non trivially.



Surgery



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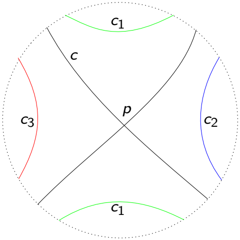

Proposition

Let c be a self-intersecting closed curve and c_1, c_2, c_3 obtained by performing surgery. Then for any k

$$\ell_k^\rho(c_1) \leq \ell_k^\rho(c) \quad \text{and} \quad \ell_k^\rho(c_2) + \ell_k^\rho(c_3) \leq \ell_k^\rho(c).$$

Part V

Systolic inequalities

Systole and entropy

We want to relate the following.

1. The k-systole length of S

$$L_k^{\rho}(S) := \min\{\ell_k^{\rho}(c) \colon c \in \mathcal{CG}(S)\}.$$

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Lemma

- *There exists a simple k-systole.*
- *$h_k^\rho(S)$ is finite for every k.*

A first systolic inequality

Theorem (M.-Zhang)

There exists a constant $C \in \mathbb{R}^+$ depending only on the topology of S such that

$$h_k^\rho(S) L_k^\rho(S) \leq C$$

for any ρ Hitchin and $k \in \{1, 2, \ldots, n-1\}$.

Short pants decomposition

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Definition

The k-panted systole length is

$$K_k^\rho(S') := \min\{\ell_k^\rho(c) \colon c \in \mathcal{CG}(S') \text{ is not a multiple of a geodesic in } \mathcal{P}_k^\rho(S')\}$$

A stronger systolic inequality

Theorem (M.-Zhang)

There exists a constant C' depending only on the topology of S' such that for any ρ Hitchin, k and $\mathcal{P}_k^\rho(S')$

$$\frac{1}{4}\log 2 \leq h_k^\rho(S')K_k^\rho(S') \leq C'\Big(\log(4)+1+\log\left(1+\frac{\sqrt{5}+1}{2}\frac{K_k^\rho(S')}{L_k^\rho(S')}\right)\Big)$$

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Remark

$$C' = 400 \cdot 3^{3g-3+n} + 23.$$

A criterion for the degeneration of the entropy

Corollary

Let $\{\rho_j : \Gamma \to \mathrm{PSL}(n_j, \mathbb{R})\}_{j=1}^{\infty}$ *be a sequence of Hitchin representations, let* $k_j \in \{1, \ldots, n_j - 1\}$ *and assume* $L_{k_j}^{\rho_j}(S') \geq \varepsilon > 0$ *for all* j*. Then, the following are equivalent:*

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1. $\lim_{j\to\infty} h_{k_j}^{\rho_j}(S') = 0$,
2. *Up to passing to a subsequence and up to passing to the action of* $\mathrm{Mod(S)}$*, there is a collection* $\mathcal{D}$ *of pairwise non-intersecting simple closed geodesics in* S' *so that*

$$\sup_j \max\{\ell_{k_j}^{\rho_j}(c) : c \in \mathcal{D}\} < \infty, \textit{ and}$$

$$\lim_{j\to\infty} \min\{\ell_{k_j}^{\rho_j}(c) : c \in \mathcal{CG}(S' \setminus \mathcal{D}) \textit{ is non-peripheral}\} = \infty.$$

Main steps of the proof

1. Fix an ideal triangulation $\mathcal{T}_k^\rho$ "adapted" to the minimal pants decomposition $\mathcal{P}_k^\rho$.
2. Combinatorial description of closed geodesics via their intersection with $\mathcal{T}_k^\rho$ and $\mathcal{P}_k^\rho$.
3. Lower bounds for the length of a closed geodesic in terms of: crossing the pants decomposition, winding along closed geodesics, crossing the triangulation.

Grazie