Varieties of Characters

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- Ex: $\mathrm{SL}(n, \mathbb{C})$ or $\mathrm{SL}(n, \mathbb{R})$ or $\mathrm{SU}(n)$ or $\mathrm{SO}(n)$

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- One notable exception is the case when there are no relations r_i, that is, when Γ is a free group.
- However, this does show that

$$\mathrm{Hom}(\Gamma, G) \cong \{(g_1, ..., g_r) \in G^r \mid r_i(g_1, ..., g_r) = \mathbf{1},\ i \in I\} \subset G^r$$

is a subvariety of the smooth variety G^r.

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- If G is real algebraic, then the character variety is semi-algebraic.
- If G is compact, then the character variety is compact and is just the usual orbit space of $\mathrm{Hom}(\Gamma, G)$.
- If G is complex reductive, then the character variety is identified with the GIT quotient.

Why care?

If you Google any of the following key words, you will find that the study of character varieties *at least* touches the following topics:

flat G-bundles, G-Higgs bundles, holomorphic vector bundles, (G, X)-structures, Mirror symmetry, String vacua, Yang-Mills connections, Convex Cocompactness, knot invariants, Geometric Langlands, Quantization, Spin Networks, A-polynomial, hyperbolic manifolds

Examples

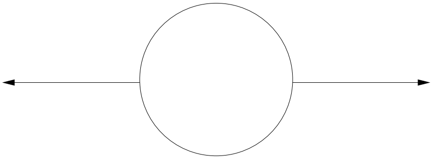


Figure : $\mathfrak{X}_{\mathbb{Z}}(\mathrm{SL}(2,\mathbb{R}))$

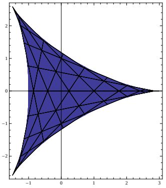


Figure : $\mathfrak{X}_{\mathbb{Z}}(\mathrm{SU}(3))$

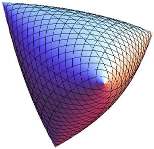


Figure : $\mathfrak{X}_{\mathbb{Z}\star\mathbb{Z}}(\mathsf{SU}(2))$

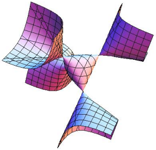
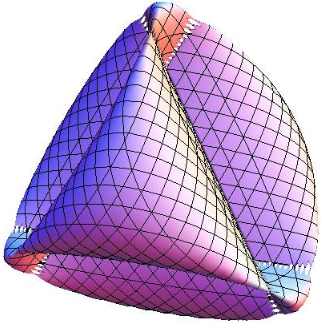


Figure : $\mathfrak{X}_{\mathbb{Z}\star\mathbb{Z}}(\mathsf{SL}(2,\mathbb{C}))(\mathbb{R})$ with boundary value 2.

$\mathfrak{X}_{\mathbb{Z}^3}(\mathrm{SU}(2))$ is a 3 dimensional orbifold with 8 singularities; each locally $\mathcal{C}_{\mathbb{R}}(\mathbb{R}P^2)$.



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One of the theorems they showed in this paper is that the space of $SL(2, \mathbb{C})$-characters was a variety...hence the term.

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- For example, $\mathfrak{X}_{F_2}(\mathrm{SO}(2, \mathbb{C}))$ is homotopic to $S^1 \times S^1$ while $\mathcal{C}h_{F_2}(\mathrm{SO}(2, \mathbb{C}))$ is homotopic to S^2.

Let $\mathfrak{Y}_r = \mathfrak{gl}(n, \mathbb{C})^{\times r} /\!\!/ \mathrm{SL}(n, \mathbb{C})$. Then in 1976 Procesi proved:

Theorem (Procesi)

$\mathbb{C}[\mathfrak{Y}_r]$ is generated by the invariants $\mathrm{tr}(\mathbf{X}_{i_1}\mathbf{X}_{i_2}\cdots\mathbf{X}_{i_k})$, where $\mathbf{X}_j$ are generic matrices.

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Therefore, $\mathbb{C}[\mathfrak{X}_\Gamma(\mathrm{SL}(n, \mathbb{C}))]$ is generated by the invariants $\mathrm{tr}(\mathbf{X}_{i_1}\mathbf{X}_{i_2}\cdots\mathbf{X}_{i_k})$, where $\mathbf{X}_j$ are generic *unimodular* matrices.

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More generally, $\mathcal{T}_\Gamma(G)$ coincides with $\mathbb{C}[\mathfrak{X}_\Gamma(G)]$ for any Γ and G equal to $\mathrm{SL}(m,\mathbb{C})$, $\mathrm{Sp}(2m,\mathbb{C})$, or $\mathrm{SO}(2m+1,\mathbb{C})$; a result of Lawton and also Sikora.

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How do they relate in general?

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- However, π is onto and $Out_T(G)$ is finite.

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- Since G is a reductive subgroup of $\mathrm{SL}(n,\mathbb{C})$, ρ and $\alpha\rho$ are completely reducible representations in $\mathrm{SL}(n,\mathbb{C})$ and, therefore, their $\mathrm{SL}(n,\mathbb{C})$-conjugation orbits are closed in $\mathrm{Hom}(F_2, G)$.

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- Thus, since ρ is irreducible, this implies that $\alpha : G \to G$ coincides with conjugation of G by some element of $\mathrm{SL}(n,\mathbb{C})$.

Proof

- Let $\alpha \in Aut_T(G)$ and consider irreducible $\rho : F_2 \to G$, so $\rho(F_2)$ is Zariski dense in G.
- Since $\mathbb{C}[\mathfrak{X}_{F_2}(\mathrm{SL}(n,\mathbb{C}))] = \mathcal{T}_{F_2}(\mathrm{SL}(n,\mathbb{C}))$ and ρ and $\alpha\rho$ have the same character, they coincide in $\mathfrak{X}_{F_2}(\mathrm{SL}(n,\mathbb{C}))$.
- Since G is a reductive subgroup of $\mathrm{SL}(n,\mathbb{C})$, ρ and $\alpha\rho$ are completely reducible representations in $\mathrm{SL}(n,\mathbb{C})$ and, therefore, their $\mathrm{SL}(n,\mathbb{C})$-conjugation orbits are closed in $\mathrm{Hom}(F_2, G)$.
- Since there is a unique closed orbit in each equivalence class in $\mathfrak{X}_{F_2}(\mathrm{SL}(n,\mathbb{C}))$, ρ and $\alpha\rho$ are conjugate in $\mathrm{SL}(n,\mathbb{C})$.
- Thus, since ρ is irreducible, this implies that $\alpha : G \to G$ coincides with conjugation of G by some element of $\mathrm{SL}(n,\mathbb{C})$.
- Now, $Out_T(G)$ is the epimorphic image of $\mathcal{N}(G)/G$ which is finite by a result of Vinberg. $\square$

Theorem (Lawton-Sikora, 2016)

Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a connected reductive group, and Γ a finitely generated group. Then:

1. $Out_T(G)$ *acts freely on* $\mathfrak{X}^i_\Gamma(G)$.
2. $\varphi^i : \mathfrak{X}^i_\Gamma(G)/Out_T(G) \to \mathcal{C}h^i_\Gamma(G)$ *is a finite, birational bijection.*
3. *If* $\mathfrak{X}^i_\Gamma(G)/Out_T(G)$ *is normal, then* φ^i *is a normalization map.*

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So although character varieties are not varieties of characters in general, we do have:

$$[\mathfrak{X}^i_\Gamma(G)/Out_T(G)] = [\mathcal{C}h^i_\Gamma(G)].$$

Corollaries

Corollary

Under the assumptions of the last Theorem, we have:

1. *If* $\mathfrak{X}_\Gamma(G)/Out_T(G)$ *is irreducible and contains an irreducible representation, then* φ *is a birational morphism.*
2. *If* $\mathfrak{X}_\Gamma(G)/Out_T(G)$ *is normal and contains an irreducible representation, then* φ *is a normalization map.*
3. φ *is a normalization map for* Γ *a free group of rank* $r \geq 2$*.*
4. φ *is a normalization map for* Γ *a genus* $g \geq 2$ *surface group and* $G = \mathrm{SL}(m, \mathbb{C})$ *or* $\mathrm{GL}(m, \mathbb{C})$*,* $m \geq 1$*.*
5. φ *is a normalization map for* Γ *a free abelian group of rank* $r \geq 1$ *and* $G = \mathrm{SL}(m, \mathbb{C})$ *or* $\mathrm{Sp}(2m, \mathbb{C})$*,* $m \geq 1$*.*

Corollary

Let F_r be a free group of rank $r \geq 2$. If G is a connected reductive subgroup of $\mathrm{SL}(n, \mathbb{C})$ such that all simple factors of the Lie algebra of the derived subgroup $[G, G]$ have rank 2 or more, then $\mathfrak{X}^{sm}_{F_r}(G)$ is étale equivalent to $\mathcal{C}h^{sm}_{F_r}(G)$.

Thank you!

- References are at

  http://arxiv.org/a/lawton_s_1.

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