Coxeter Mapping Classes

AMS Sectional Meeting – Bowdoin 2016

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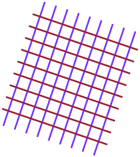
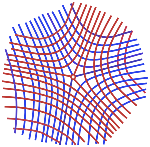
I. Space of pseudo-Anosov mapping classes

Pseudo-Anosov mapping classes

Let S be a compact oriented surface of finite type, $\phi : S \to S$ a mapping class. By Nielsen-Thurston: ϕ is periodic, reducible or pseudo-Anosov.

Definition: $\phi : S \to S$ is *pseudo-Anosov* if there is a pair of ϕ-invariant transverse measured singular foliations on S so that for some $\lambda > 1$, the action of ϕ stretches one measure by λ and contracts the other by $1/\lambda$. The *(geometric) dilatation* $\lambda_{\text{geo}} = \lambda$ depends only on the mapping class ϕ.





Space of pseudo-Anosov mapping classes

Consider the space of all pseudo-Anosov mapping classes.

$$\mathfrak{P} = \{(S, \phi) \mid S \text{ compact oriented surface}, \phi \in \text{Mod}(S) \text{ is pA}\}$$

- Flow equivalence classes: *mapping classes with related dynamics*
  - polynomial invariants: *Alexander and Teichmüller polynomials*
  - homological and geometric dilatations
- Thurston's Fibered Cone theory, gives identifications of flow equivalence classes with the rational points on open polygons.
- Operations: *Hopf plumbing, Murasugi sum, Train-track folding automata.* ("poset structure")
- Relations with combinatorics: *finite directed graphs, generalized Perron-Frobenius matrices.*

II. Mixed-sign Coxeter systems

Mixed-sign Coxeter systems

Coxeter graph: $\Gamma = (\mathcal{V}, \mathcal{E})$ finite connected graph with no self-edges or double edges.
$\mathcal{V} = \{v_1, \ldots, v_k\}$ ordered vertices. $A =$ adjacency matrix.

Reflection group. Coxeter group W acts on $\mathbb{R}^{\mathcal{V}}$, preserving the bilinear form $B = 2I_{\mathfrak{s}} - A$, and is generated by $s_1, \ldots, s_k$, where each s_i is a reflection through the hyperplane perpendicular to v_i:

$$s_i(v_j) = v_j - \mathfrak{s}(v_i) B_{i,j} v_i.$$

Coxeter element $\omega = s_1 \cdots s_k$.

Coxeter polynomial $C(x) = \det(xI - \omega)$.

Positive (Classical) Case

Assume $\mathfrak{s}$ is positive.

- Interlacing lemma implies the spectral radius of ω is monotone increasing with Γ.
- Presentation for Coxeter group

$$\mathcal{W} = \langle s_1, \ldots, s_k \mid (s_i s_j)^{m_{i,j}} = 1 \rangle$$

  where $m_{i,j} = 2 + A_{i,j} - \delta_{i,j}$. This is a quotient of the Artin group

$$\mathcal{A} = \langle \beta_1, \ldots, \beta_k \mid [\beta_i \beta_j]_{(m_{i,j})} = [\beta_j \beta_i]_{(m_{i,j})} \rangle.$$

- The minimum spectral radius larger than one among essential elements of $\mathcal{W}$ is attained by a Coxeter element and overall minimum Lehmer's number ≈ 1.17628 is attained by E_{10}. [McMullen]
- Nice divisibility properties of Coxeter polynomials [Gross-H-McMullen]

Mixed sign Case

Cons:

- No interlacing theorem.
- Presentation of Coxeter group???
- Coxeter elements don't necessarily have the minimum spectral radius.

Pros:

- Nice divisibility properties. [Billet]

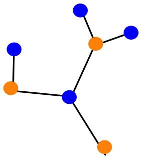
Bipartite Alternating case [H-Liechti]:

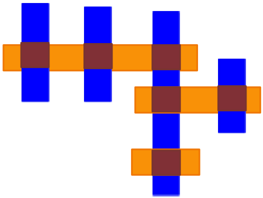
- Interlacing theorem
- Minimum spectral radius $\frac{3+\sqrt{5}}{2}$ attained by alternating A_2.
- All eigenvalues of ω are either on the unit circle or real.

II. Constructing Mapping classes from Coxeter graphs.

Coxeter mapping classes for sign-labeled fat trees (also works for bipartite graphs)

Build surface S from tree Γ:





Surface: S = dual configuration of rectangular strips with ends identified.

Mapping class: Central lines of the rectangular stripcs define simple closed curves $\gamma_1, \ldots, \gamma_k$ on S one for each vertex in V_Γ. Let

$$\phi = \delta_1^{\pm 1} \cdots \delta_k^{\pm 1},$$

where δ_i are positive Dehn twists along γ_i.

Coxeter mapping classes

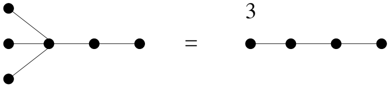
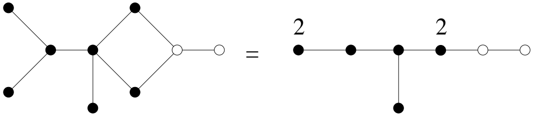
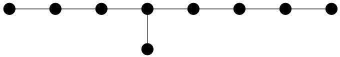
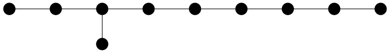
- (Howlett, H) The Alexander polynomial $\Delta(x)$ of (S, ϕ) and the Coxeter polynomial $C(x)$ of Γ satisfy $\Delta(x) = C(-x)$.
- (Thurston, H, Leininger) If Γ is positive, then either
  - Γ is spherical: A_n, D_n, E_6, E_7, E_8 and (S, ϕ) is periodic;
  - Γ is affine: $\widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$ and (S, ϕ) is reducible; or
  - (S, ϕ) is an orientable pseudo-Anosov mapping class.
- (McMullen) If Γ is positive, smallest dilatation = Lehmer's number, K = (-2,3,7)-pretzel.)
- (H, Liechti) If Γ is an alternating tree, then (S, ϕ) is orientable, pA, and is the monodromy of an alternating link. (Smallest dilatation = (golden mean)^4, K = figure 8.)
- (Liechti) Smallest dilatation alternating fibered links in each genus is obtained by alternating A_n.

Cannot get small dilatation mapping classes from positive or alternating graphs...

But you can from other kinds of mixed sign graphs!

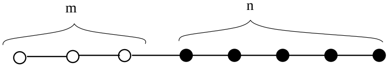
Minimum dilatation orientable p-A mapping classes

(Lanneau-Thiffeault)

genus	mixed-sign Coxeter graph
2	
3	
4	
5	

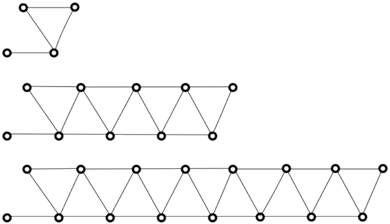
Infinite families

(Brinkman, Kin-H) As $(m, n) \to \infty$, $\lambda \to 1$



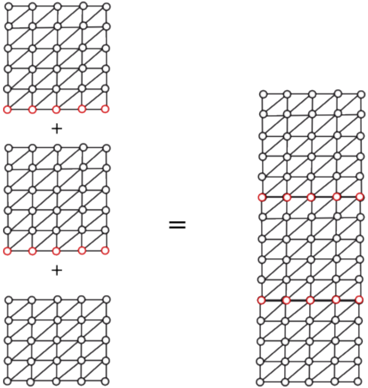
(H)

$$|\chi(S_\Gamma)| \to \infty \quad \text{and} \quad \lambda^{|\chi(S_\Gamma)|} \to \left(\frac{3+\sqrt{5}}{2}\right)^2 .$$



Periodic Coxeter mapping class

Key point: Lots of extra periodic Coxeter graphs.



Are these (together with the classical ones) all the periodic mixed-sign Coxeter graphs?

Flow Equivalence Classes

In progress...

- (Billet-Liechti) Algorithm to compute Alexander and Teichmueller polynomial for the flow equivalence classes of alternating Coxeter trees.
- "taking sums" of flow equivalence classes using Murasugi sum.

Thank You!