Bisectors in the Bidisk

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- This is a rank two symmetric space : *flats* are of the form "geodesic x geodesic" and thus two-dimensional.
- *Bisectors*, which are equidistant hypersurfaces between pairs of points, display interesting geometric properties.
- We will see that generically, the bisector between the pair of points $\mathbf{x}, \mathbf{y}$ determine that pair uniquely.

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- Therefore :

$$\mathrm{Isom}(\mathbf{B}) = \left(\mathrm{Isom}(\mathbf{H}^2) \times \mathrm{Isom}(\mathbf{H}^2)\right) \times \langle\sigma\rangle$$

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- I use sans-serif font to denote a point in $\mathbf{B}$ and the same letter, in regular font, to denote its coordinates in $\mathbf{H}^2$:

$$\mathbf{x} = (x_1, x_2)$$
$$\mathbf{y} = (y_1, y_2) \text{ etc.}$$

Your basic bisector

- Given $\mathbf{x}, \mathbf{y} \in \mathbf{B}$, their *bisector* is the set of points equidistant to $\mathbf{x}$ and $\mathbf{y}$:

$$\mathrm{E}(\mathbf{x}, \mathbf{y}) = \{\mathbf{p} \in \mathbf{B} \ | \ \rho(\mathbf{x}, \mathbf{p}) = \rho(\mathbf{y}, \mathbf{p})\}$$

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- Rewrite as follows :

$$\sqrt{d^2(x_1, p_1) + d^2(x_2, p_2)} = \sqrt{d^2(y_1, p_1) + d^2(y_2, p_2)}$$

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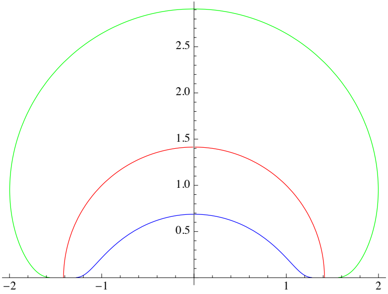
- This motivates the following definition...

Definition

The square hyperbola of level k is :

$$\mathrm{SH}_k(x,y) = \{p \in \mathbf{H}^2 \mid d^2(x,p) - d^2(y,p) = k\}$$



Remark : Square hyperbolae are relevant in the generic case. If $x_1 = y_1$ or $x_2 = y_2$, we are in the non-generic case where $\mathrm{E}(\mathbf{x}, \mathbf{y})$ is the product of the hyperbolic plane with a geodesic in the hyperbolic plane.

- A generic bisector admits a natural foliation by products of square hyperbolae :

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- Given $\mathbf{x},\mathbf{y}\in\mathbf{B}$ and any $p\in\mathbf{H}^2$, there exists a point $\mathbf{p}\in\mathbf{B}$ lying on the bisector $\mathrm{E}(\mathbf{x},\mathbf{y})$ with p as one of its coordinates. For example, given $p_1\in\mathbf{H}^2$, choose any $p_2\in \mathrm{SH}_{-L_{x_1,y_1}(p_1)}(x_2,y_2)$.

Theorem

Let $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v} \in \mathbf{B}$ *such that* $\mathrm{E}(\mathbf{x}, \mathbf{y}) = \mathrm{E}(\mathbf{u}, \mathbf{v})$. *Assume that the bisector is generic. Then :*

$$\{\mathbf{x}, \mathbf{y}\} = \{\mathbf{u}, \mathbf{v}\}.$$

Outline of proof

- **Step 1 :** if two bisectors are equal, then they must share the same leaves up to re-parametrization.

Lemma

Let $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v} \in \mathbf{B}$ *such that* $\mathrm{E}(\mathbf{x}, \mathbf{y}) = \mathrm{E}(\mathbf{u}, \mathbf{v})$*. Assume that the bisector is generic. Then for every* $k \in \mathbb{R}$*, there exists* $m \in \mathbb{R}$ *such that :*

$$\mathrm{SH}_k(x_1, y_1) \times \mathrm{SH}_{-k}(x_2, y_2) = \mathrm{SH}_m(u_1, v_1) \times \mathrm{SH}_{-m}(u_2, v_2).$$

Furthermore, if $k = 0$*, then* $m = 0$*.*

In this case, call the points $x_1, y_1, u_1, v_1 \in \mathbf{H}^2$ *SH-related*.

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- Except for the point at the intersection of μ and $\mathrm{SH}_m(u, v)$, there are always two points on $\mathrm{SH}_m(u, v)$ which are at equal distance from $\mathrm{SH}_0(u, v)$.
- Therefore y must belong μ, and by symmetry, so must x.

- **Step 3 :** Given an ordered set of distinct points $\Omega = \{x, y, u, v\} \subset \mathbf{H}^2$, set :

$$\begin{aligned} \Phi_\Omega : \mathbf{H}^2 \setminus \mathrm{SH}_0(u,v) &\longrightarrow \mathbb{R} \\ p &\longmapsto \frac{L_{x,y}(p)}{L_{u,v}(p)}. \end{aligned} \tag{1}$$

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- Lemma : Assume that $\{x, y\}$ and $\{u, v\}$ are SH-related. Then Φ_Ω is constant.
- Proposition : If x, y, u, v are distinct and collinear points, such that $\mathrm{SH}_0(x, y) = \mathrm{SH}_0(u, v)$, then Φ_Ω is not constant.