Self-Similar Boundary Layer Flows

Introduction

In this project we implement classical similarity methods for solving the governing equations for mass, momentum and energy in laminar boundary layer flow over a flat plate. The mass and momentum equations are discussed in virtually all books dealing with viscous fluid flow and will only be highlighted below. We begin by computing the velocities in the boundary layer using a numerical solution to the Blasius equation, and then use the velocity field that we have computed as input to two fundamental forced convection calculations also expressed in terms of similarity variables. In the first we find the temperature in a flat plate boundary layer in which the temperature of the plate is fixed at a value different than the freestream value. In the second we include viscous dissipation in the boundary layer as the source of heat and apply an adiabatic boundary condition at the surface. The wall temperature we compute in the second case is known as the adiabatic wall temperature. The solution of the Blasius equation will be implemented using 4th order Runge-Kutta integration, while for the temperature calculations we select a finite-difference method. The formulation presented here follows the exposition given in Chow (1979) very closely, but fills in a few implementation details.

Learning Objectives

By the time you are finished with this exercise you will have:

1. Reviewed the laminar, boundary layer equations for flow over a flat plate,

2. Reviewed the similarity transformation that allows the set of two PDE's governing continuity and x-momentum in the boundary layer to be collapsed into a single ODE,

3. Applied 4th order Runge-Kutta to the solution of a 3rd order ODE (the Blasius equation) and implemented boundary conditions,

4. Used the Runge-Kutta results for the velocity profile and a finite-difference representation of the similarity form of the energy equation to solve for (a) the temperature profile in a laminar boundary layer on a heated plate and (b) the temperature in a boundary layer where viscous dissipation is important,

5. Applied two different types of thermal boundary conditions: fixed temperature and adiabatic,

6. Used "canned" software to solve a tridiagonal system of linear equations,

7. Displayed and verified computed results,

8. Observed the wildly different heat transfer behavior of three representative fluids.

Background

For the classical scenario of 2-D, laminar boundary layer flow on a flat plate as depicted below (See Figure 1), the governing mass, momentum and energy conservation equations are written (Chow, 1979):

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho\, u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0, \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \tag{2}$$

and

$$\rho\, c_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{3}$$

The last term in Equation 3 represents viscous dissipation (the conversion of mechanical energy into heat) in the boundary layer and is the heat source in the second of the two forced convection problems to be solved later. For steady conditions, uniform properties and negligible pressure gradient, these equations reduce to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2}, \tag{5}$$

$$\rho\, c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2, \tag{6}$$

The value of u, the streamwise velocity, increases from 0 at y = 0, the plate surface, to U_∞ at y = $\delta(x)$, the boundary layer edge. If the shapes of the velocity profiles are to be similar from station to station along the plate, then we expect the u velocity to be a function of x and y only in the combination

$$\eta = \frac{y}{\delta} \tag{7}$$

Here δ is the local boundary layer thickness.

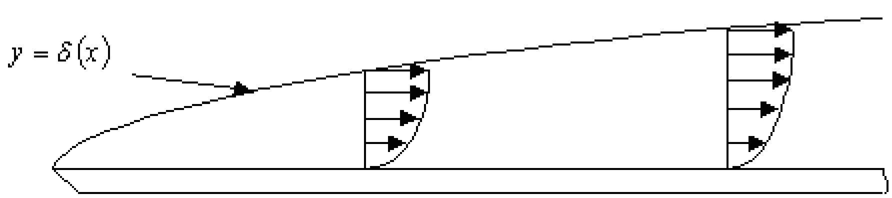

Figure 1. Schematic for laminar, flat plate boundary layer

A detailed scaling analysis (see e.g., Gebhart, 1971) shows that:

$$\frac{\delta}{x} \approx \frac{1}{\left(\frac{\rho U_\infty x}{\mu}\right)^{\frac{1}{2}}} \tag{8}$$

That is, a laminar boundary layer grows as the square root of the coordinate in the streamwise direction (x). Thus,

$$\eta = \frac{y}{\left(\frac{\upsilon x}{U_\infty}\right)^{\frac{1}{2}}} = y\left(\frac{U_\infty}{\upsilon x}\right)^{\frac{1}{2}} = \frac{y}{x}\sqrt{Re_x} \tag{9}$$

At this point we define a streamfunction ψ, such that in the usual fashion $\frac{\partial \psi}{\partial y} = u$ and $\frac{\partial \psi}{\partial x} = -v$. When these definitions are substituted in Equation 4 above, we note that continuity is satisfied automatically. Then we replace $\psi(x,y)$ by a function f of the similarity variable η :

$$\psi = \sqrt{\upsilon x U_\infty}\, f(\eta) = \upsilon \sqrt{Re_x}\, f(\eta). \tag{10}$$

Then u, v, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$, and $\frac{\partial^2 u}{\partial y^2}$ can all be expressed in terms of the similarity parameter. For example,

$$u = \frac{\partial \psi}{\partial y} = \sqrt{\upsilon x U_\infty}\, \frac{\partial f(\eta)}{\partial y} = \sqrt{\upsilon x U_\infty}\, \frac{\partial f(\eta)}{\partial \eta}\frac{\partial \eta}{\partial y} \tag{11}$$

$$= \sqrt{\upsilon x U_\infty}\, f' \frac{1}{\sqrt{\upsilon x / U_\infty}} = U_\infty f'.$$

Once all terms in Equation 5 are replaced in terms of similarity variables and the equation is simplified, we get the well-known equation solved by Blasius in 1908:

$$ff'' + 2f''' = 0 \tag{12}$$

When transformed, the boundary conditions become $f = f' = 0$ at $\eta = 0$ and $f' = 1$ at $\eta = \infty$. The first and third come from u = 0, u = U_∞ at the surface and the freestream, respectively. The other comes from v = 0 at the surface.

The introduction of the similarity variable η means that instead of the coupled system of PDE's (Equations 4 and 5), we have a single third order, non-linear ODE to solve. In preparation for solving Equation 12 using 4th order Runge-Kutta (Hoffman, 1992), we first break that equation into a system of three first-order equations:

$$\frac{df}{d\eta} = p\ (= f') \tag{13a}$$

$$\frac{dp}{d\eta} = q \left(= f''\right) \tag{13b}$$

$$\frac{dq}{d\eta} = -\frac{1}{2} f\, q \;\left(= f'''\right) \tag{13c}$$

With the step size equal to $\Delta\eta$, we can then write the formulae for 4th order Runge-Kutta:

$$\Delta_1 f = \Delta\eta\, p_j \tag{14a}$$

$$\Delta_1 p = \Delta\eta\; q_j \tag{14b}$$

$$\Delta_1 q = -.5^* \Delta\eta\; f_j\, q_j \tag{14c}$$

$$\Delta_2 f = \Delta\eta\; ^*\left(p_j + .5^*\, \Delta_1 p\right) \tag{14d}$$

$$\Delta_2 p = \Delta\eta\; ^*\left(\, q_j + .5^* \Delta_1 q\right) \tag{14e}$$

$$\Delta_2 q \;= -.5^*\, \Delta\eta\; ^*\left(f_j + .5^* \Delta_1 f\right)\; ^*\; \left(\, q_j + .5^* \Delta_1 q\right) \tag{14f}$$

$$\Delta_3 f = \Delta\eta\; ^*\left(p_j \;+ .5^*\; \Delta_2 p\right) \tag{14g}$$

$$\Delta_3 p = \Delta\eta\;\; ^*\; \left(q_j + .5^* \Delta_2 q\right) \tag{14h}$$

$$\Delta_3 q = -.5^*\, \Delta\eta\; ^*\left(f_j + .5^*\, \Delta_2 f\;\right)^*\left(q_j + .5^*\, \Delta_2 q\right) \tag{14i}$$

$$\Delta_4 f = \Delta\eta\;\; ^*\; \left(p_j + \Delta_3 p\right) \tag{14j}$$

$$\Delta_4 p = \Delta\eta\; ^*\left(\, q_j + \Delta_3 q\right) \tag{14k}$$

$$\Delta_4 q = -.5^*\, \Delta\eta\; ^*\left(f_j + \;\Delta_3 f\right)^*\left(\, q_j + \Delta_3 q\right) \tag{14l}$$

With these computed changes the new values of f, p and q are found from:

$$f_{j+1} \;= f_j \;+ \left(\Delta_1 f + 2^*\Delta_2 f + 2^*\Delta_3 f + \Delta_4 f\right)/6 \tag{15a}$$

$$p_{j+1} \;= p_j \;+ \left(\Delta_1 p + 2^*\Delta_2 p + 2^*\Delta_3 p + \Delta_4 p\right)/6 \tag{15b}$$

$$q_{j+1} \;= q_j \;+ \left(\Delta_1 q + 2^*\Delta_2 q + 2^*\Delta_3 q + \Delta_4 q\right)/6 \tag{15c}$$

The above formulae are implemented in the two subroutines RUNGEK.FOR and Rungek90.for, which like much of the material above, are taken in part from Chow (1979). Many numerical methods books will provide a graphical interpretation for 2nd order Runge-Kutta, though not for 4th order.

Were all three boundary conditions given at the surface ($\eta = 0$), then it would be a simple matter to use the above formulae to march from there out to some value, say η_{max}, sufficiently large as to approximate infinity. Unfortunately one condition is given at the edge of the boundary layer rather than at the surface. Several schemes are discussed in Chow. One of these is an iterative shooting method in which successive guesses are made for the value missing at the surface (f''). That is, we take a shot using an assumed value of f'' and note the difference in value of f' at η_{max} relative to the desired value of $f' = 1.0$. We keep changing the value of $f''(0)$ by a fixed increment until we overshoot (or undershoot) the desired value. Any time that we switch from overshoot to undershoot (or vice-versa) we change the sign of the increment and divide it by 2.0. Using this "half-interval" method we eventually home in on the desired value.

Another method, also outlined by Chow, allows the solution to be found with only two shots, instead of the multiple shots used in the previous method. We assume a linear transformation such that $\eta = kz$, where k is an algebraic constant and introduce a function g such that $f(\eta) = g(z)/k$. Then it is easy to show that:

$$\frac{\partial^n f}{\partial \eta^n} = \frac{1}{k^{n+1}} \frac{\partial^n g}{\partial z^n}. \tag{16}$$

In terms of g, the Blasius equation becomes:

$$g''' + \tfrac{1}{2} g g'' = 0, \tag{17}$$

with boundary conditions $g = g' = 0$ at z = 0 and $g' = k^2$ at $z = \infty$. Then we run the calculation starting with a convenient value of $g'' = 1.0$ at z = 0; and since $f'' = \frac{1}{k^3} g''$, then $f'' = \frac{1}{k^3}$ at $\eta = 0$. A second pass using this value of $f''(0)$ will yield the correct solution. The procedure just described is implemented in the RUNGEK/Rungek90 subroutines and all that needs to be done is to call it.

Of particular interest is the value of f'' at the surface. We can write:

$$\tau_0(x) = \mu \left(\frac{\partial u}{\partial y} \right)_0 = \mu U_\infty f''(0) \left(\frac{U_\infty}{\upsilon x} \right)^{1/2}. \tag{18}$$

This local wall shear stress may be integrated over the length of the plate to get the total drag. (And if you want to "cheat", instead of using one of the two methods discussed above you could look up the formula for either the local shear stress or the drag in laminar flow on a plate to find the starting value $f''(0)$.)

Heat Transfer Calculations

We now turn our attention to the solution of Equation 6. We will solve the similarity form of this equation using finite-difference methods. Before doing so we should note that solutions for several cases under the restriction of negligible viscous dissipation are already available. For instance, for a Prandtl number ($= \frac{\upsilon}{\alpha} = \frac{c_p \mu}{k}$) of 1.0, Equations 5 and 6 are the same, as are their suitably defined boundary conditions. Thus for Pr = 1.0 the temperature solution is directly available from the velocity solution just computed. Also for cases of extremely high and low Prandtl numbers, the energy equation simplifies to allow an analytical solution (indeed these extreme cases may cause certain problems for the numerical solution we develop here for moderate Prandtl numbers). Details may be found in Gebhart (1971).

We define: $\theta = \frac{T - T_\infty}{\Delta T_{ref}}$ (19)

For the case without viscous dissipation, $\Delta T_{ref} = T_1 - T_\infty$, with T_1 being the wall temperature. With viscous dissipation $\Delta T_{ref} = U^2/2c_p$. Then the energy equation becomes:

$$\frac{d^2\theta}{d\eta^2} + \tfrac{1}{2}\Pr f \frac{d\theta}{d\eta} = -2\Pr\left(f''\right)^2, \tag{20}$$

where the term on the right hand side represents the effect of viscous dissipation. With centered differences used for both first and second derivatives, Eqn. 20 may be approximated as:

$$\frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{\left(\Delta\eta\right)^2} + \tfrac{1}{2}\Pr f\left(\eta\right)\frac{\theta_{i+1} - \theta_{i-1}}{2\Delta\eta} = -2\Pr\left(f''\left(\eta\right)\right)^2 \tag{21}$$

The values $f(\eta)$ and $f''(\eta)$ come from the previous Runge-Kutta solution of the Blasius equation. This sequential solution of the momentum and continuity equations followed by the solution of the energy equation is typical of forced convection problems. Natural (free) convection calculations (in which these equations are coupled) are typically much more difficult. The system of linear equations represented by Equation 21 is seen to be tridiagonal, since each unknown temperature (θ_i) depends only on the (unknown) temperatures of its two immediate neighbors. Such a "banded" system is readily solved using, for instance, SGTSV, the tridiagonal solver from the LAPACK family of linear equation solvers. Ample documentation is contained internally in SGTSV (Anderson et al., 1992).

Two cases are to be studied for each of three different fluids: water (Pr = 6.75), air (Pr = .71) and mercury (Pr = .044). In the first case where viscous dissipation is negligible, the temperature of the wall $\theta_1 = 1.0$, while the edge of the boundary layer is keep at $\theta_\infty = 0.0$. Whereas an η_{max} = 10.0 will be plenty sufficient for the velocity calculation ($\eta = 5.0$ is conventionally taken as the edge of the velocity boundary layer), you will need to experiment to find a suitable extent for the thermal boundary layer, particularly for the mercury. Similarly you may find a problem for high Prandtl number fluids (oils) where the thermal boundary layer is much thinner than the velocity boundary layer.

The numbering system used in SGTSV is shown schematically here:

D(1)	DU(1)								= B(1)
DL(1)	D(2)	DU(2)							= B(2)
	DL(2)	D(3)	DU(3)						= B(3)
				…	…	…			= …
							DL(IMAX-1)	D(IMAX)	= B(Imax)

Here D represents the diagonal coefficient, DU the superdiagonal coefficient and DL the subdiagonal coefficient. The values for the first row come readily from the application of the boundary condition at the wall, i.e., $\theta_1 = 1.0$, leading to D(1) = 1.0, DU(1) = 0.0 and B(1) = 1.0. The entries in the second and subsequent rows come from applying Equation 21 at the second and subsequent points away from the wall. (The index for the subdiagonal entry (DL) is, in fact, one less than the others on the same row.) The last equation comes from applying the outer boundary condition $\theta_{imax} = 0$ and as a result DL(IMAX-1) = 0.0.

With viscous dissipation the wall may be made adiabatic by making $\theta_o = \theta_2$. Here θ_o is a fictitious cell *inside* the wall; θ_2 is one increment off the wall. This value goes implicitly into Equation 21, which is used at the wall this time rather than the fixed boundary temperature applied for the first set of calculations. This time DU(1) also has a non-zero value, as do B(2) and subsequent values of B. Again the outer boundary is kept at $\theta = 0.0$ Plot up the temperature profiles for each set of three cases, i.e., you should generate a plot for no viscous dissipation (three values of Pr) and another with viscous dissipation (three values). Typical results for the second case are shown in Figure 2.

The method suggested here for incorporation of the boundary conditions into the tridiagonal system of equations makes the programming very simple, but you should check to make sure that introducing the extra equations at the ends does not lead to *scaling* problems (Chapra and Canale, 1988) due to widely differing magnitudes in the coefficients.

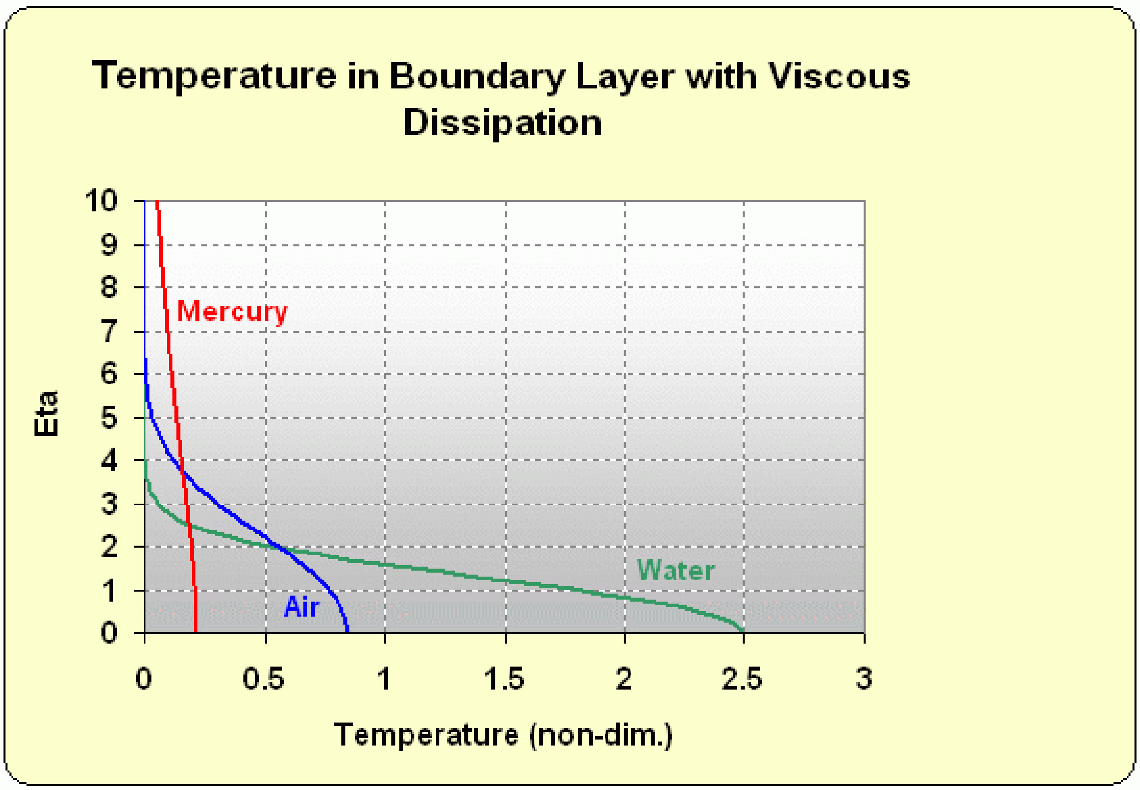

Figure 2. Temperature distribution across flat plate boundary layer with viscous dissipation

Once you have done this calculation, there are a number of things you can do with the results. With the results from the case without viscous dissipation, you can come up with a correlation for the local or average heat transfer from the plate, including the dependence on Prandtl number. With the second set of results, you can find the dependence of the recovery factor, a result of great importance in high-speed flow, on Prandtl number. The recovery factor (see e.g., Gebhart (1971) or Kays and Crawford (1993) for a complete discussion) is defined as:

$$b = \frac{T_a - T_\infty}{U_\infty^2 / 2c_p} \qquad (22)$$

Deliverables

Your writeup should at minimum include two plots, one for the case of a fixed wall temperature and no viscous dissipation; the other with viscous dissipation. Each plot should include the three fluids stipulated and you should discuss all results.

Other Things That Can Be Done with This Project (Extras)

1. Derive the boundary layer equations (Equations 4, 5 and 6).

2. Go through the arguments that result in Equation 8.

3. Express v, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial^2 u}{\partial y^2}$ in terms of the similarity variables (See the process for the u velocity component in Equation 11), so that you can fill in the steps leading to the Blasius equation (3).

4. Change the subroutine RUNGEK.FOR/Rungek90 to use shooting based on the half-interval method (top of fourth page) to satisfy the boundary condition at infinity.

5. Compare the values you find for f, f', and f'' with tabulated values.

6. Compare your value for the skin friction on a plate (Equation 18) with the accepted value.

7. Derive the similarity form of the energy equation (Equation 20).

8. Using your computed results for the case without viscous dissipation and with a fixed value of temperature at the wall, determine a correlation for the local and mean Nusselt number ($\frac{h_x x}{k}$ and $\frac{hL}{k}$, respectively) for heat transfer to or from a flat plate.

9. For Pr = 1.0 compare the velocity profile with the temperature profile for the case with fixed wall temperature but without viscous dissipation.

10. Verify the validity (and in fact the necessity) of the correlations given e.g. in Gebhart (1971) or Kays and Crawford (1993) for extreme Prandtl numbers (high and low).

11. Use your results from the second part (with viscous heating and an adiabatic condition at the surface) to predict the temperature of the skin of an aircraft flying at a Mach number of 2.5.

Verification

Numerical results for f, f' and f'' for the Blasius equation are tabulated in nearly every fluid mechanics and heat transfer book (see, e.g., Incropera and DeWitt, 2002). Heat transfer results with and without viscous dissipation are available both as temperature vs. position data (Kays and Crawford, 1993) and in the form of overall correlations (Incropera and DeWitt, 2002).

References

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