Laminar, Thermal Entry Length Heat Transfer

Introduction

In this exercise we implement a numerical solution for thermal entry length heat transfer in laminar, internal flow. In a thermal entry length problem, the velocity profile is already fully developed when a change in the wall thermal boundary condition is introduced. In addition to flows involving an unheated starting length (where the velocity profile is developing), a thermal entry length solution may also be a good approximation for high Prandtl number fluids (oils) for which the velocity field is established more quickly than the temperature field. Thermal entry length problems for laminar flows are commonly solved using analytical methods (and indeed, were solved that way over a century ago by Graetz), but the scheme to be developed here will prove to be easy to implement and amenable to a wide variety of boundary conditions. In addition this project serves as an excellent introduction to the so-called *finite-volume* method.

The thermal entry length problem as presented here is one of three related problems (See, e.g., Kays and Crawford, 1993.) The first, in order of increasing difficulty, involves fully developed velocity and temperature profiles. (In the case of the temperature profile, "fully developed" means that the shape, although not the absolute level, is no longer a function of the axial coordinate. Otherwise there would be no heat transfer.) For this case the energy equation reduces to an ordinary differential equation for the temperature as a function of radius, and the non-dimensional heat transfer coefficient (the Nusselt number) works out to be a constant (the well-known values of 3.66 for a fixed wall temperature and 4.36 for a uniform heat flux). The thermal entry length scenario discussed below is the second of the three related problems. The third involves velocity and temperature profiles both developing and is considerably more difficult because partial differential equations must be solved for the radial and axial velocity distributions along with the temperature field.

Learning Objectives

By the time you are done with this exercise you will be able to:

1. Recognize a parabolic equation and explain why a once-through or marching approach is applicable,
2. Write a control-volume energy balance in cylindrical geometry,
3. Describe the advantage of the finite-volume approach over the finite-difference approach for cylindrical geometries,
4. Define the consistency property,
5. Explain the similarities and differences between this problem and the problem of transient conduction in one dimension,
6. Explain the stability implications of this problem,
7. Implement the solution of a tridiagonal system of linear equations,
8. Compute and verify the Nusselt number.

Background - Conservation

Rather than simply taking the non-negligible terms in the governing partial differential equation and finite-differencing them, we will work directly with conservation of energy for an annular control volume; that is, we will simply apply the conservation statement "in - out = 0" to a representative small volume. As seen in the schematic of a representative control volume (Figure 1), the only transport mechanisms we will consider are radial conduction in through the inner radius, radial conduction out through the outer radius and axial advection of heat into and out of the ends.

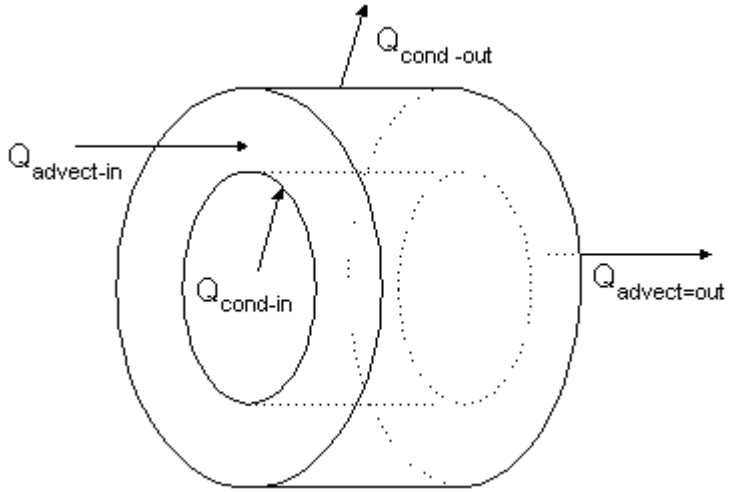


Figure 1. A representative control volume

So that the physical significance of each term will be readily apparent, we will work at first in dimensional variables. Bearing in mind that our volumes are in fact toroidal, Figure 2 shows the variables in a slice through the computing region. Only the upper half is shown.

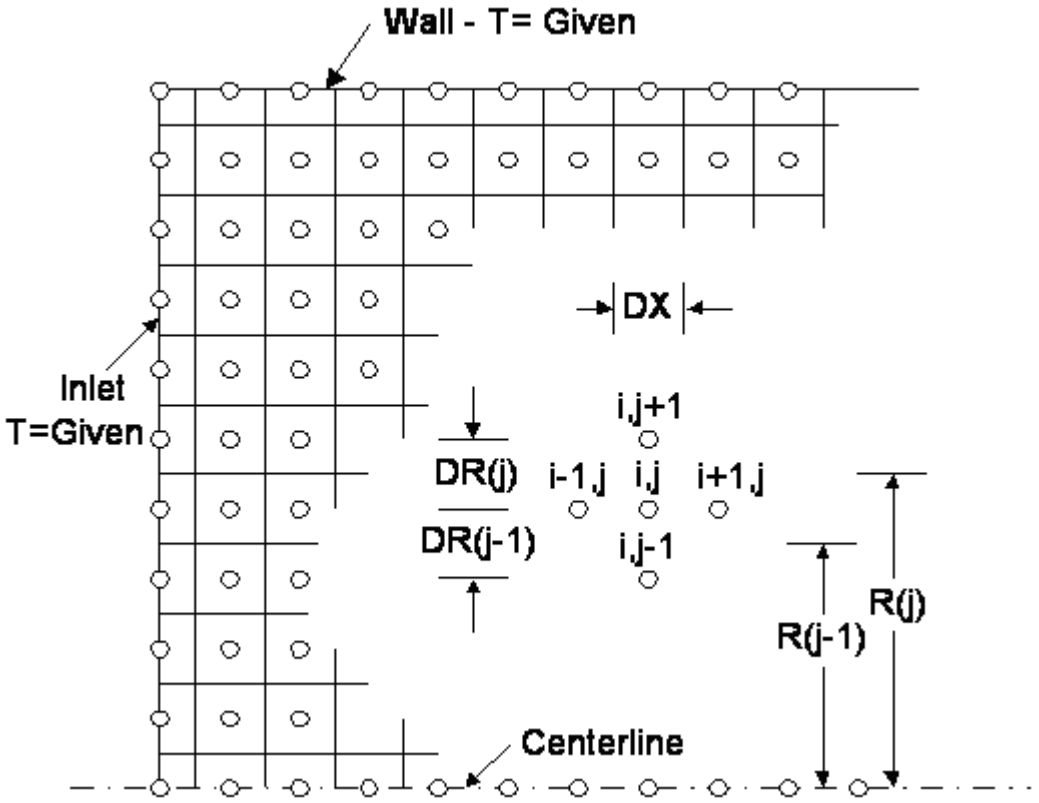
Notice that R(j) indicates the outer radius of the annular volume whose temperature is $T_{i,j}$; R(j-1) is its inner radius. Although you will probably take the radial grid spacing as uniform, DR(j) is taken to represent the spacing between $T_{i,j}$ and $T_{i,j+1}$. The innermost control volume (at the centerline) is a solid cylinder of radius DR(1)/2. The energy balance is then written as:

$$- k 2\pi R_{j-1} dx \frac{T_{i,j}-T_{i,j-1}}{DR_{j-1}} - (-) k 2\pi R_j dx \frac{T_{i,j+1}-T_{i,j}}{DR_j} \qquad (1)$$

$$+ \rho c_p U_j \pi \left[R_j^2 - R_{j-1}^2\right] T_{i-1,j} - \rho c_p U_j \pi \left[R_j^2 - R_{j-1}^2\right] T_{i,j} = 0.0$$

Here, of course, the first two terms represent conduction through the inner and outer radial face of the toroidal control volume, respectively. The other terms represent transport by advection (heat carried along with the moving fluid) into and out the ends of the control volume. Thus, axial conduction is neglected; this approximation is generally valid except for very slow flow of highly conductive fluids (liquid metal). (The criterion as to whether or not axial conduction is important is based on the magnitude of the Peclet number ($Pe = RePr = \frac{UD}{\alpha}$, where $\alpha = \frac{k}{\rho c_p}$ and is the thermal diffusivity.) This particular assumption is very important because it permits the "once-through" marching procedure we intend to implement. (If that term were not negligible, we would have a boundary condition to satisfy at the outlet of our computing region. Without it, we don't.) Also because the velocity field is fully developed at the point

where heating begins, there is no radial velocity and therefore no advective transport through the radial faces. Thus the process we model here in convection at its simplest: radial conduction moves heat away from the walls into the main flow where advection sweeps it away.



Background - Consistency

If we note that the area of the ends of our control volumes, written above as $\pi\left[R_j^2 - R_{j-1}^2\right]$ for finite spacing, becomes $2\pi R\Delta R$ for small grid spacing, then it is clear that once the grid spacing is shrunk to zero size the above numerical approximation is entirely equivalent to the governing partial differential equation:

$$\frac{u}{\alpha}\frac{\partial T}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial T}{\partial r}, \qquad (2)$$

as it ought to. Thus the proposed finite-volume scheme has the desired “consistency" property.

Background - Stability

The particular differencing scheme depicted in Equation 1 and which is to be implemented in code demands some explanation. If we disregard the r, 1/r and the fact that u = u(r) in the above partial differential equation, we note that it has exactly the form of the transient, one-dimensional conduction equation, i.e.,

$$\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \tag{3}$$

A very simple forward-time, central-space (explicit) finite-difference approximation of this parabolic (in time) equation is:

$$\rho c_p \left[\frac{T_i^{n+1} - T_i^n}{\Delta t} \right] = k \frac{T_{i-1}^n - 2T_i^n + T_{i+1}^n}{(\Delta x)^2} \tag{4}$$

The very simplest stability analysis, i.e., the requirement of positivity of the central coefficient, shows that this explicit approximation has a time step restriction of the form: $r = \frac{\alpha \Delta t}{(\Delta x)^2} \le \frac{1}{2}$. For the transient conduction equation this translates to a time step restriction. For our situation in which the equation is parabolic in space rather than in time, this translates to a restriction on the marching step size of the form $\frac{\alpha \Delta x}{u(\Delta r)^2} \le \frac{1}{2}$. Near the wall the local velocity and possibly the radial grid spacing get very small, and this marching step size restriction would more than likely become impossible to meet. If violated, we would expect to find an instability starting at the nodes nearest the wall and spreading into the interior. Thus, as with transient conduction problems when the allowable time step is too restrictive, we use an implicit solution for the unknown temperatures at the new station. Beginning at the first column of unknowns, we will solve a tridiagonal system of linear equations for the temperature distribution at each axial position before marching on to the next one. As with transient conduction problems, other slightly more complicated techniques, e.g., the Crank Nicolson method, would give better accuracy. In particular the Crank Nicolson scheme, in which the average at the current and previous stations are used in the radial derivatives, results in the first derivative being centered in space rather than the being "one-sided" as it is above.

Implementation

To make this solution as general as possible, we would like to do the calculation in non-dimensional form. To do that we introduce the non-dimensional variables: $r' = \frac{r}{r_o}$, $u' = \frac{u}{u_m}$, and $T' = \frac{T - T_w}{T_i - T_w}$. Here r_o is the tube radius, u_m is the mean velocity, T_w is the wall temperature, and T_i is the inlet temperature. With these quantities defined the appropriate scaling for x comes out to be $x' = \frac{x\alpha}{u_m r_o^2} = \frac{x}{Re\ Pr\ r_o}$. The inverse of this non-dimensional length is commonly referred to as the *Graetz* number. (But in comparing results with other sources make sure it is defined the same way. Some sources use the diameter rather than radius as the scaling dimension.) With the introduction of this scaling (and dropping primes from here on), the non-dimensional equation looks exactly like Equation 2, but without the α. Furthermore, it is seen that the solution will be independent of the Reynolds and Prandtl number. The results tabulated in all heat transfer books for entry length heat transfer give a good idea of the range of non-dimensional axial length you need to consider. You will need to simply set the inlet temperature to 1.0 and the wall temperature to 0.0 (at least for the fixed wall temperature case); run the radius r from 0.0 to 1.0. For a laminar (Hagen-Poiseuille) flow, the velocity in non-dimensional form is given by $u = 2(1 - r^2)$, i.e., it is a parabola.

You will note that in Figure 2 the centerline of the pipe is taken to be at the bottom. That way $\frac{\partial T}{\partial r}$ is taken in the upward direction. To start with, take about 11 grid points in the radial direction so that you have 10 unknowns (the wall temperature is fixed). Note that with the differencing scheme proposed in Equation 1 above, the proper treatment of the node at the centerline is readily apparent (Hint - R_{j-1} becomes equal to 0.0), whereas the equivalent partial differential equation (Equation 2) gives no guidance at all as to what to do at r = 0. Alleviating quandaries of this sort is the major advantage of the finite volume method (Equation 1) over the more traditional finite-difference method, which is what one would apply to Equation 2.

About 20 stations in the flow (x) direction should suffice to start, so that the temperature would be dimensioned T(20,11). (In fact, where a marching solution is employed, only the current and previous radial temperature distributions really would need to be stored.) With $T_{i-1,j}$ known and $T_{i,11}$ also known, then at each axial station you will have a tridiagonal system to solve for the unknown temperature profile. The LAPACK subroutine SGTSV does this efficiently. When a station is completed you simply step downstream and repeat the calculation. NOTE that SGTSV destroys the coefficients passed to it; thus they must be reset each time before it is called.

The numbering scheme used for the coefficients and right hand sides passed to SGTSV is as shown below:

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D(1)   DU(1)                                                    = B(1)
DL(1)  D(2)   DU(2)                                             = B(2)
       DL(2)  D(3)                                              = B(3)
                …      …                                        = …
                                  DL(Jmax-1)   D(Jmax)          =B(Jmax)
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The vector of diagonal coefficients and the vector of right hand sides are of length JMAX; the super and sub-diagonal coefficient vectors are of length JMAX-1. At the centerline (I=1) the subdiagonal is seen to be 0.0. With the grid used in the sketch, JMAX could be taken as either 10 or 11. If JMAX = 10 is used, then the fixed value at the wall is incorporated into the right hand side for the cell next to the wall. Otherwise if JMAX = 11 is used, then the last equation is DL(10) = 0.0; D(11) = 1.0 and B(11) = 0.0. (Again you will want to be sure that your tridiagonal system of equations is scaled well.)

Processing Results

The "bottom line" for this calculation is the local Nusselt number as a function of distance from the beginning of the heated section. It may be found from the following:

$$Nu_x = \frac{h_x D}{k} \tag{5}$$

Here,

$$h = \frac{q''}{\Delta T} = \frac{-k \left.\frac{\partial T}{\partial y}\right|_{r = r_o}}{T_x - T_m}. \tag{6}$$

Here T_x is the wall temperature (a constant as this problem stands) and T_m is the mean temperature defined for an incompressible flow as:

$$T_m = \frac{2}{u_m r_o^2} \int_0^{r_o} u\, T\, r\, dr \qquad (7)$$

This quantity will have to be evaluated numerically (as a summation) at each axial station. Thus,

$$Nu_x = \frac{2}{T_x - T_m} \left. \frac{\partial T}{\partial r} \right|_{r=1} \qquad (8)$$

where $\left. \frac{\partial T}{\partial r} \right|_{r=1}$ is also evaluated numerically from the computed results. This value should approach the fully developed value (3.658 for this boundary condition) as we proceed downstream. Now the temperatures at the current station become the upstream values, and we move on downstream to do the calculation at the next station. The axial variation of Nu_x and T_m should be plotted.

Some Extras

1. This program may be easily extended to the case of a specified heat flux at the wall.

2. The case of a uniform (plug) velocity profile is a trivial extension. The plug flow approximation may be applicable to low Prandtl number fluids, in which the temperature profile is set up more rapidly than the velocity profile.

3. Both viscous dissipation (frictional heating) and volumetric heating (due to chemical or nuclear reactions) may be included easily.

4. Annular geometries subjected to any combination of thermal boundary conditions at the inner and outer radii are easily handled.

5. The axial variation of surface temperature can be added easily. Duhamel's theorem, Stieltjes integrals, etc. (Kays and Crawford, 1993) are not needed. Axial variation of heat flux can be readily handled providing item 1 above has been implemented.

6. Make a contour plot of temperature.

Verification

The analytical solution for the basic problem outlined above (constant temperature wall boundary condition) was given in 1883 by Graetz (Eckert and Drake, 1972). Analytical solutions are similarly available for most of the additional scenarios listed above. For the constant wall temperature case Kays and Crawford (1993) give the following values of local Nusselt number as a function of non-dimensional distance. Similar values are available for the uniform heat flux case.

x'	Nu_x
0.001	12.80

0.004	8.03
0.01	6.00
0.04	4.17
0.08	3.77
0.10	3.71
0.20	3.66
∞	3.66

References

Eckert, E.R.G. and Drake, R. M.,Jr., *Analysis of Heat and Mass Transfer*, McGraw-Hill (1972).

Kays, W.M. and Crawford, M.E., *Convective Heat and Mass Transfer*, 3rd Ed., McGraw-Hill, New York (1993).

Shah, R.K. and London, A.L., *Laminar Flow Forced Convection in Ducts*, Academic Press, New York (1978).