Flow in a Duct Having an L-shaped Cross-Section

Introduction

In this exercise you will apply one (or more) iterative methods for solving the large system of linear equations arising from the finite-difference discretization of an elliptic equation. In addition you will get to review and use the concept of the friction factor for flow in pipes by using your numerical results to find the value for a particular cross-section.

Background

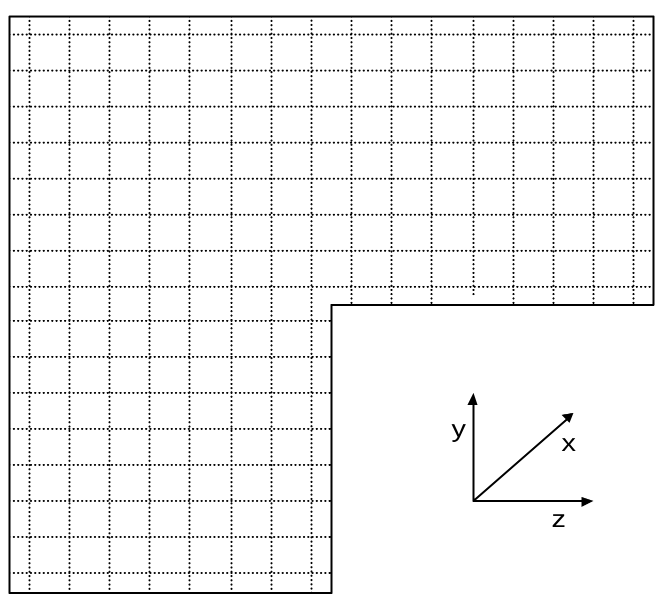
Consider steady, fully-developed, laminar, incompressible, unidirectional flow in a duct having the L-shaped cross-section shown below. Under these restrictions the flow direction (x) component of the equations of motion reduces to:

$$0 = -\frac{dp}{dx} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

The above equation simply states the balance between the pressure difference driving the flow and the viscous restraining forces.

Implementation

Your first job is to discretize the above PDE using second-central differences, solve the resulting system of linear equations to find the flow field and plot isovels (contours of constant u velocity). Note that since the equation is linear, it does not matter what value you use for $\frac{dp}{dx}$ or the viscosity (μ) or their ratio. Use the relatively coarse mesh shown below and a simple pointwise iterative scheme, e.g., Gauss-Seidel, to start.



You might use parameter statements so that you can change mesh sizes readily later. Note that the grid points are considered to be at the centers of the cells plotted and that half-cells are used along edges and quarter cells at the corners. Of course, the value of u is known (a boundary condition) at all those partial cells anyway.

Use your computed results to determine the friction factor for laminar flow in this geometry. To the accuracy you can expect on a grid this coarse, you will not be able to draw any conclusions, but you should get close to the value you expect for a cylindrical, smooth tube in laminar flow (Hagen-Poiseuille flow). There are several ways to do this calculation. The easiest is to note that between your input parameters and your computed results, you have everything you would need if you were doing the analogous experiment in the laboratory. You need to assume the friction factor has the form f = Constant/Reynolds# and your job is to find the constant. For Hagen-Poiseuille flow in a circular tube the constant is 64. This value is derived in virtually all undergraduate fluid mechanics books.

Extra

1. Implement a faster method for solving the elliptic equation; e.g., alternating-direction-implicit (ADI), multi-grid or conjugate gradient, and use a finer mesh. In this case you might work with the files SGTSV (the tridiagonal solver from LAPACK), CYCLEC or CONJUG, respectively. Recompute the friction factor on a finer mesh.

2. Consider a non-Newtonian fluid, the viscosity of which is a function of the shear rate. For this case the viscosity must be retained within the derivatives above and updated locally as the iteration proceeds. The more general form of the equation above is:

$$0 = -\frac{dp}{dx} + \frac{\partial}{\partial y}\mu\frac{\partial u}{\partial y} + \frac{\partial}{\partial z}\mu\frac{\partial u}{\partial z}$$

Here the appropriate place at which to evaluate μ is at the interfaces between cells, rather than at cell centers where the velocity is assumed defined. One of the simplest non-Newtonian viscosity models is the power law model (Burmeister, 1993, Appendix B).

$$\tau = \mu_o \left|\dot{\gamma}\right|^{n-1}\dot{\gamma}$$

A value of n = 1 corresponds to a Newtonian fluid; a value less than one corresponds to a *shear-thinning* fluid and a value in excess of unity corresponds to a *shear-thickening* fluid. For the simple flow being studied here, the magnitude of the strain rate is given by:

$$\left|\dot{\gamma}\right| = \left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right]^{1/2}$$

Run your program for values ranging from n = 1/5 to n = 5. Plot velocity contours and comment.

References

Fletcher, C.A.J., *Computational Techniques for Fluid Dynamics Vol. I Fundamental and General Techniques*, Springer-Verlag, Berlin (1991).

Kays, W.M. and Crawford,M.E., *Convective Heat and Mass Transfer*, 3rd Ed., McGraw-Hill, New York (1993).

Burmeister, L.C., *Convective Heat Transfer*, 2nd Ed., Wiley-Interscience, New York (1993).

Software Provided

SGTSV - LAPACK subroutine for solving tridiagonal systems of linear equations. Use in implementing ADI method.

CYCLEC - Program implementing the multigrid scheme for the Poisson equation on a rectangular region.

CONJUG - Program implementing the conjugate gradient scheme for the Poisson equation on a rectangular region.

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