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Three models for the homotopy theory of homotopy theories

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What is a homotopy theory?

We take the point of view that a homotopy theory is just a simplicial category.

Definition 1. A (small) *simplicial category* is a category with a set of objects and a simplicial set of morphisms between any two objects.

We will denote the category of simplicial categories $\mathcal{SC}$.

Why should a simplicial category be a homotopy theory?

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Recall: Given a model category $\mathcal{M}$, there is associated to it $\mathrm{Ho}(\mathcal{M})$, its homotopy category.

There is also its *simplicial localization* $\mathcal{LM}$ which is a simplicial category encoding the known homotopy-theoretic information about $\mathcal{M}$.

Let $\pi_0\mathcal{LM}$ denote the category of components of $\mathcal{LM}$, namely the category with objects those of $\mathcal{LM}$ and morphisms given by

$$\mathrm{Hom}_{\pi_0\mathcal{LM}}(x,y) = \pi_0\mathrm{Hom}_{\mathcal{LM}}(x,y).$$

Then $\pi_0\mathcal{LM} = \mathrm{Ho}(\mathcal{M})$.

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So, the simplicial localization is a kind of enriched version of the homotopy category.

Ignoring set-theoretic difficulties, one can take the simplicial localization of any category with a choice of weak equivalences, even without the additional structure of a model category.

We would like a notion of weak equivalence of simplicial categories which is a simplicial analogue to the notion of equivalence of categories.

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Definition 2. A map $f : \mathcal{C} \to \mathcal{D}$ of simplicial categories is a *DK-equivalence* if:

- $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(fx, fy)$ is a weak equivalence of simplicial sets for any objects x, y of $\mathcal{C}$, and
- $\pi_0 f : \pi_0\mathcal{C} \to \pi_0\mathcal{D}$ is an equivalence of categories.

Up to DK-equivalence, any simplicial category can be obtained by taking the simplicial localization of a category with weak equivalences. Therefore, it makes sense to consider simplicial category to be a homotopy theory.

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Simplicial categories as the first model for the homotopy theory of homotopy theories

To talk about the "homotopy theory of homotopy theories," we would like to have a model category structure on the category of all (small) simplicial categories with appropriate weak equivalences.

Theorem 3. *There is a model category structure on the category $\mathcal{SC}$ of all small simplicial categories in which the weak equivalences are the DK-equivalences.*

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The problem is that these weak equivalences of simplicial categories are difficult to work with, in that it is difficult to tell whether a given map of simplicial categories is actually a DK-equivalence. It would be helpful to have another model category with nicer weak equivalences in which to obtain information about homotopy theories.

We would like to use "complete Segal spaces" instead.

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The second model: complete Segal spaces

Definition 4. A *simplicial space* is a simplicial object in the category of simplicial sets, namely a functor $\mathbf{\Delta}^{op} \rightarrow \mathcal{S}Sets$.

Definition 5. A *Segal space* is a Reedy fibrant simplicial space W such that the Segal maps

$$\varphi_k : W_k \rightarrow \underbrace{W_1 \times_{W_0} \cdots \times_{W_0} W_1}_{k}$$

are weak equivalences of simplicial sets for $k \geq 2$.

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We can talk about Segal spaces much like we talk about (simplicial) categories:

- "objects": the set $W_{0,0}$
- "morphism space" $\mathrm{map}_W(x, y)$ = the fiber over (x, y) of the map $W_1 \to W_0 \times W_0$

Can also define "compositions" and "homotopy equivalences."

There is also a "homotopy category" $\mathrm{Ho}(W)$.

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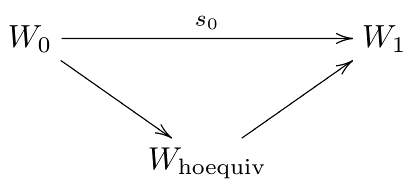
There is a space of homotopy equivalences

$$W_{\mathrm{hoequiv}} \subseteq W_1.$$

Note that the degeneracy map $s_0 : W_0 \to W_1$ factors through W_{hoequiv}:



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We are now able to define the objects that we want to work with.

Definition 6. A *complete Segal space* W is a Segal space such that the map $W_0 \to W_{\mathrm{hoequiv}}$ is a weak equivalence of simplicial sets.

What do complete Segal spaces have to do with simplicial categories?

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Rezk defines a functor taking any simplicial category to a complete Segal space. If $\mathcal{C}$ is a discrete category, then the corresponding complete Segal space $N\mathcal{C}$ is given by

$$(N\mathcal{C})_n = \mathrm{nerve}(\mathrm{iso}\ \mathcal{C}^{[n]}).$$

So, the space in degree 0 is the nerve of the maximal subgroupoid of $\mathcal{C}$, and the rest of the category gets encoded in degree 1.

If $\mathcal{C}$ is a simplicial category, the functor is more complicated, but the essential idea is the same.

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To compare the homotopy theory of complete Segal spaces to the homotopy theory of simplicial categories, we would like to have an appropriate model category structure on complete Segal spaces.

There is no model category structure on the category of complete Segal spaces, however, since it does not have all limits and colimits.

We will use a model category structure obtained by localizing the Reedy model category structure on simplicial spaces with respect to a map.

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Theorem 7. *(Rezk) There is a model category structure $\mathcal{CSS}$ on the category of all simplicial spaces such that:*

1. *the fibrant objects are the complete Segal spaces and all objects are cofibrant,*
2. *a weak equivalence $f : W \to Z$ between Segal spaces is a "DK-equivalence":*
   - *$map_W(x, y) \to map_Z(fx, fy)$ is a weak equivalence of simplicial sets for any $x, y \in W_{0,0}$,*
   - *$Ho(W) \to Ho(Z)$ is an equivalence of categories,*
3. *a weak equivalence $f : W \to Z$ between complete Segal spaces is a levelwise weak equivalence of simplicial sets.*

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The model category $\mathcal{CSS}$ is nicer to work with than $\mathcal{SC}$ since

- the objects are just diagrams of spaces, and
- the weak equivalences (at least between complete Segal spaces) are easy to identify.

We would like a Quillen equivalence between the model categories $\mathcal{SC}$ and $\mathcal{CSS}$.

However, there does not seem to be an appropriate adjoint pair between the two categories.

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We will try to find an intermediate category which is Quillen equivalent to each of the two model categories.

To motivate the objects in this intermediate category, recall that the nerve of a simplicial category $\mathcal{C}$ is a simplicial space which looks like

$$Ob(\mathcal{C}) \Leftarrow Mor(\mathcal{C}) \Lleftarrow Mor(\mathcal{C}) \times_{Ob(\mathcal{C})} Mor(\mathcal{C}) \cdots$$

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The third model: Segal categories

Definition 8. A *Segal precategory* X is a simplicial space such that X_0 is a discrete space.

Definition 9. A *Segal category* X is a Segal precategory such that the Segal map $\varphi_k : X_k \to \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{k}$ is a weak equivalence of simplicial sets for $k \geq 2$.

Note that the nerve of a simplicial category is a Segal category.

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Just as we did with Segal spaces, we can talk about the objects and morphism spaces of a Segal category, as well as its homotopy category.

Again, we cannot have a model category structure on the category of Segal categories, so we need to work in a larger category. In this case, we will use the category of Segal precategories.

Fact: There is a “localization” functor L taking any Segal precategory X to a Segal category LX.

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Theorem 10. *There is a model category structure $\mathcal{S}e\mathcal{C}at_c$ on the category of Segal precategories such that*

1. *the weak equivalences (again "DK-equivalences") are the maps $f : X \to Y$ such that*
   - $map_{LX}(x, y) \to map_{LY}(fx, fy)$ *is a weak equivalence of simplicial sets for any $x, y \in (LX)_0$, and*
   - $Ho(LX) \to Ho(LY)$ *is an equivalence of categories, and*
2. *the fibrant objects are the Reedy fibrant Segal categories, and*
3. *all objects are cofibrant.*

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Quillen Equivalences

Now we want an adjoint pair of functors

$$I : \mathcal{S}e\mathcal{C}at_c \rightleftarrows \mathcal{CSS} : R$$

The left adjoint functor $I : \mathcal{S}e\mathcal{C}at_c \to \mathcal{CSS}$ is just the inclusion functor from Segal precategories to simplicial spaces.

What is the right adjoint?

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Given a simplicial space W, take the pullback at each level of the diagram of simplicial spaces:

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
W_2 & \longrightarrow & W_0 \times W_0 \times W_0 & \longleftarrow & W_{0,0} \times W_{0,0} \times W_{0,0} \\
\Downarrow\Downarrow\Downarrow & & \Downarrow\Downarrow\Downarrow & & \Downarrow\Downarrow\Downarrow \\
W_1 & \longrightarrow & W_0 \times W_0 & \longleftarrow & W_{0,0} \times W_{0,0} \\
\Downarrow\Downarrow & & \Downarrow\Downarrow & & \Downarrow\Downarrow \\
W_0 & \longrightarrow & W_0 & \longleftarrow & W_{0,0}
\end{array}
$$

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The resulting simplicial space will be a Segal precategory which is DK-equivalent to the original W. If W was a complete Segal space, then the pullback will be a Segal category.

Theorem 11. *The adjoint pair*

$$I : \mathcal{S}e\mathcal{C}at_c \rightleftarrows \mathcal{CSS} : R$$

is a Quillen equivalence.

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We now need an adjoint pair

$$F : \mathcal{S}e\mathcal{C}at_c \rightleftarrows \mathcal{SC} : R.$$

The right adjoint functor $R : \mathcal{SC} \to \mathcal{S}e\mathcal{C}at_c$ is just the nerve functor.

The left adjoint is a "rigidification" functor, resulting in a simplicial space X such that

$$X_k \to \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{k}$$

to be an isomorphism of simplicial sets, rather than a weak equivalence.

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The problem is that this adjoint pair is not a Quillen pair. The left adjoint of a Quillen pair needs to preserve cofibrations and therefore cofibrant objects. Unlike in $\mathcal{S}e\mathcal{C}at_c$, not all objects in $\mathcal{SC}$ are cofibrant.

Thus, we need a second model category structure on the category of Segal precategories which has fewer cofibrations. We will also need it to be Quillen equivalent to $\mathcal{S}e\mathcal{C}at_c$.

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Theorem 12. *There is a model category structure* $\mathcal{S}e\mathcal{C}at_f$ *on the category of Segal precategories such that*

1. *the weak equivalences are the same as those of* $\mathcal{S}e\mathcal{C}at_c$*,*
2. *the fibrant objects are levelwise fibrant Segal categories, and*
3. *the identity functors*

$$\mathcal{S}e\mathcal{C}at_f \rightleftarrows \mathcal{S}e\mathcal{C}at_c$$

give a Quillen equivalence.

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Theorem 13. *The adjoint pair*

$$F : \mathcal{S}e\mathcal{C}at_f \rightleftarrows \mathcal{SC} : R$$

is a Quillen equivalence.

Thus, we have a chain of Quillen equivalences

$$\mathcal{SC} \leftrightarrows \mathcal{S}e\mathcal{C}at_f \rightleftarrows \mathcal{S}e\mathcal{C}at_c \rightleftarrows \mathcal{CSS}.$$