Multipermutation Codes in the Ulam Metric

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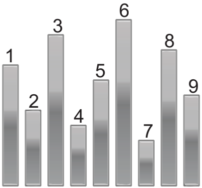
Summary

- Novel rank modulation multipermutation codes (MPCs) in Ulam metric
- Codes correcting translocation and deletion errors
- Highlight connection between MPCs in the Ulam and Hamming metrics
- Capacities or bounds for MPCs in both metrics
- Constructions using Steiner systems, BIBDs, and interleaving
- Efficienct decoding algorithms

Rank Modulation for Flash Memory

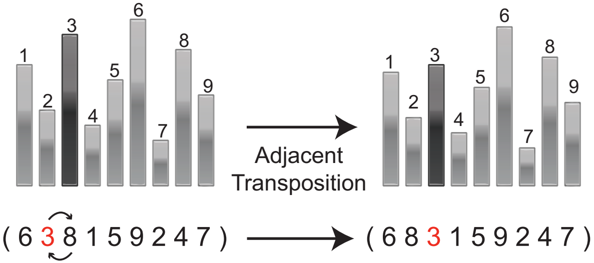
- Rank modulation for flash memory was proposed by Jiang et al. [2008] for dealing with over-injection and charge leakage.
  - In an array of cells, each cell stores a charge level.
  - Information stored in relative values of charge levels.
  - Data encoded as permutations in blocks of cells.



Permutation: (6 3 8 1 5 9 2 4 7)

Errors in Rank Modulation: Adjacent Transpositions

- Retention errors (charge leakage), cycling errors, and write-disturb errors often modeled as adjacent transpositions.



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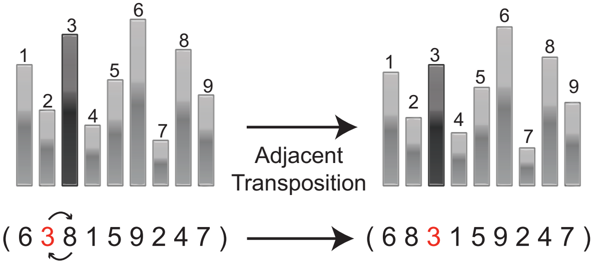

- Correcting t adjacent transposition $\Longleftrightarrow$ min Kendall tau distance $\geq 2t+1$.

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Errors in Rank Modulation: Translocations

- Increasing number of charge levels to increase capacity leads to larger charge fluctuations relative to gap between charge levels.

Farnoud, Skachek, Milenkovic, Transactions on Information Theory, May 2013

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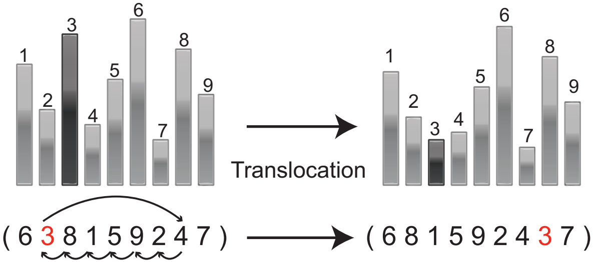

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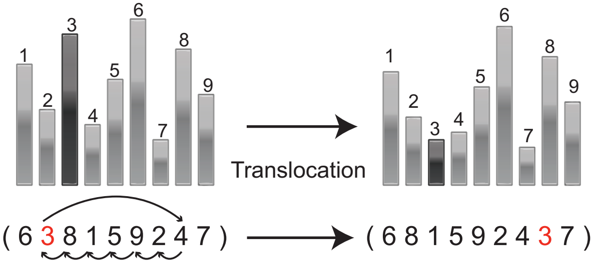

- Correcting t translocations $\iff$ min Ulam distance $\geq 2t + 1$.
- Ulam distance = length – length of longest common subsequence (LCS).

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Errors in Rank Modulation: Deletions

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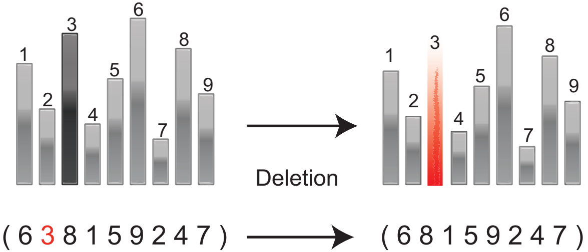

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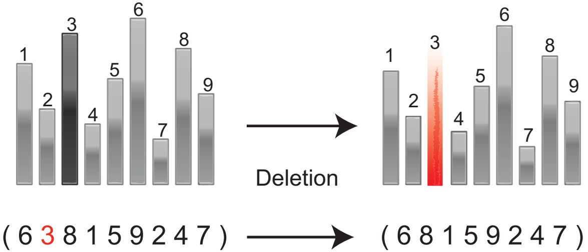

- Correcting t deletions $\iff$ min Ulam distance $\geq t+1$.

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- Multipermutation (MP): a rearrangement of elements of a multiset: $[2, 1, 1, 2]$ for the multiset $\{1, 1, 2, 2\}$.

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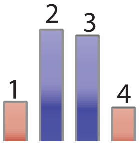

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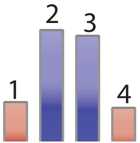

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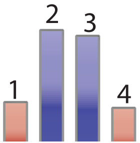

- r-regular MPs: each element (rank) appears r times.
- Beneficial since the number of possible ranks is limited.

Multipermutation Codes II

- In the literature:
  - Multipermutation re-write codes [En Gad'12]
  - Multipermutation codes in Chebyshev metric [Shieh'10,'11]
  - Multipermutation codes in Kendall tau metric [Buzaglo'13] [Sala'13]
  - Multipermutation codes in Hamming metric [Luo'03] [Ding'05] [Huczynska'06] [Chu'06]. Aka, constant composition codes, frequency permutation codes.
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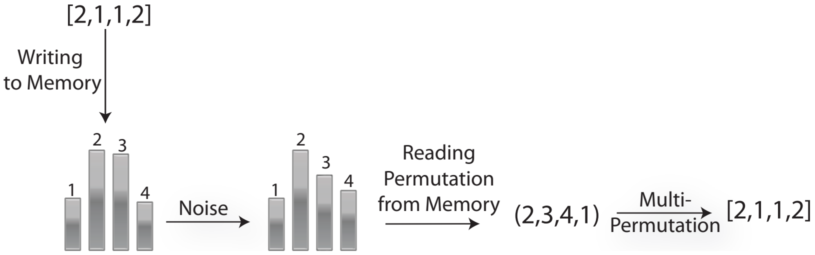
- This work: codes in the Ulam metric for correcting translocation and deletion errors.
- We consider permutations and multipermutations simultanously by considering equivalence classes of multipermutations.

Multipermutation Codes III



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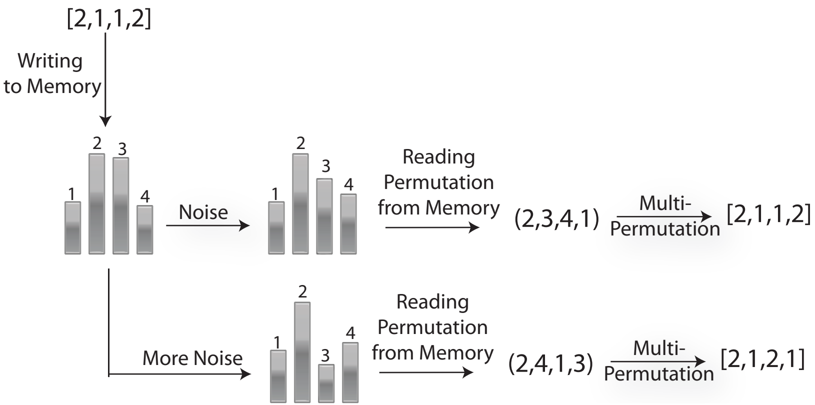
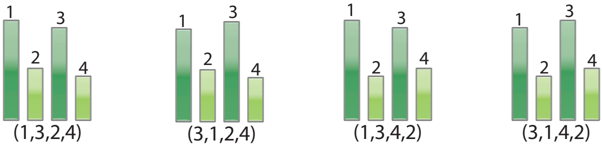


Permutations and Multipermutations

- Same information (same MP): $[1, 2, 1, 2]$:



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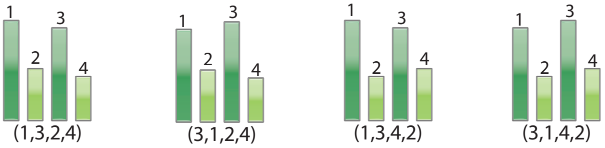

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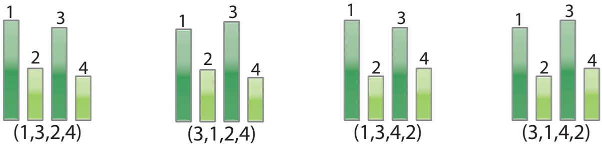

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- $R_r(\pi)$: equivalence class of π

$$R_2(1,3,2,4) = \{(1,3,2,4),(3,1,2,4),(1,3,4,2),(3,1,4,2)\}.$$

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- An r-regular MP code of length n, MPC(n, r), is a subset C of $\mathbb{S}_n$ such that if $\pi \in C$, then $R_r(\pi) \subseteq C$.
- Size of C is the number of equivalence classes it contains.

- An MPC(n, r) has minimum Ulam distance d if for all π and σ not in the same equivalence class, $d_u(\pi, \sigma) \geq d$.

Multipermutation Codes in Ulam and Hamming Metrics

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- We present construction of Ulam codes using codes in Hamming metric.

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- In a given metric, let $A(n, r, d)$ denote the maximum size of an MP code of length n, regularity r, and minimum distance d.

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Theorem [FM2014]

The capacity of multipermutation codes in the Hamming metric with parameters r and d, with $\rho = \lim_{n\to\infty} \frac{\ln r}{\ln n}$ and $\delta = \lim_{n\to\infty} \frac{d}{n}$, is given by

$$\mathcal{C}_H(r, d) = (1 - \rho)(1 - \delta).$$

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$$\left(1-2\rho\right)\left(1-\delta\right)\leq\mathcal{C}_U\left(r,d\right)\leq\left(1-\rho\right)\left(1-\delta\right).$$

Code Construction: Almost-disjoint Sets

Lemma

Let C be an MPC(n, r), and $2t < r$. If for all $\pi, \sigma \in C$, each rank of π and σ are either identical or have less than $r - 2t$ elements in common, then C can correct t translocation errors.

- Simple example for $t = 1$, $r = 6$, $n = 12$ (in MP form)

$$
\begin{aligned}
C = \{ & [1,1,1,1,1,1,2,2,2,2,2,2], \\
& [1,1,1,2,2,2,1,1,1,2,2,2], \\
& \quad \dots \\
& [2,2,2,2,2,2,1,1,1,1,1,1]\}.
\end{aligned}
$$

The intersection between each two ranks is of size $3 < 4 = r - 2t$.

Steiner Systems and BIBDs

- A k-(n, r, λ)-*design* is a family of r-subsets of a set X of size n, each called a *block*, such that every k-subset of X appears in exactly λ blocks.

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- A *Steiner system* $S(k, r, n)$ is a k-$(n, r, 1)$-design.
- A Balanced incomplete block design (BIBD) with parameters (n, r, λ) is a 2-(n, r, λ)-design.

Code Construction: Resolvable Steiner Systems

Proposition

If a resolvable Steiner system $S(k, r, n)$ exists, then for odd $d \leq r - k + 1$, there exists an MPC(n, r) with min Ulam distance d, of size

$$\frac{\binom{n-1}{k-1}}{\binom{r-1}{k-1}} \left(\frac{n}{r}\right)!.$$

- Proof outline:
  - The blocks in a class of the Steiner system are assigned as the elements of the ranks of the multipermutation.
  - Each two blocks have less than k elements in common.
  - Size of code follows from the number of classes and the fact that blocks can be assigned to ranks arbitrarily.

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  - Example: $r = 3$, each row is a block, each table is a class.

1	2	3
4	5	6
7	8	9

1	4	7
2	5	8
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Proposition

Suppose that r is an odd prime. Then, there is an MPC(r^2, r) with minimum Ulam distance $r - 2$ and size $(r + 1)r!$.

Code Construction: Interleaving I

- Let $\circ$ denote interleaving:
  - $(1,3,2) \circ (6,4,5) \circ (8,7,9) = (1,6,8,3,4,7,2,5,9)$.

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- Assume
  - A partition $\{P_1, \ldots, P_r\}$ of $[n]$ into sets of equal size.
  - Let C_i, $i \in [r]$, be permutation codes of minimum Ulam distance $d \leq n/r$ over P_i.
  - Construct C by interleaving codewords of C_i.

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- Assuming optimal component codes, if $\lim \frac{rd}{n}=0$, then C is capacity achieving.

Code Construction: Interleaving II

- Let $\circ_r$ denote interleaving blocks of r elements:
  - $(\underline{1,3},\underline{4,2}) \circ_2 (\underline{6,7},\underline{8,5}) \circ_2 (\underline{12,10},\underline{9,11}) =$
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- Assume
  - n/r even, $d \leq r$, $P = \left[\frac{n}{2}\right]$, and $Q = [n] \backslash P$.
  - C'_1 is an MPC$(\frac{n}{2}, r)$ with min Ulam distance d over P
  - C_1 is an MPC$(\frac{n}{2}, r)$ with min Hamming distance d over Q.

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The code $C = C_1' \circ_r C_1$ is an MPC(n, r) with minimum Ulam distance d.

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- Assume
  - n/r even, $d \leq r$, $P = \left[\frac{n}{2}\right]$, and $Q = [n] \setminus P$.
  - C_1' is an MPC($\frac{n}{2}, r$) with min Ulam distance d over P
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- Assuming optimal Hamming codes, C is capacity achieving (for $d \leq r$).

Thank you!