Rank Modulation Codes for Translocation Errors in Flash Memories

F. Farnoud

with V. Skachek and O. Milenkovic

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Permutations for coding

Permutations for coding:

- Store (or transmit) a permutation instead of an arbitrary sequence

History:

- Introduced by: Slepian [1965]
- With substitution errors: Blake [1974]
- With adjacent-swap errors: Chadwick and Kurtz [1969]
- Application to power-line communications, substitution errors: Vinck [2000]
- Application to flash memories, adjacent-swap errors: Jiang et al. [2008]: Rank modulation

Motivation: Rank Modulation

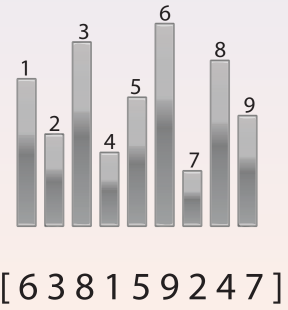
Rank modulation for flash memories: proposed for dealing with charge leakage in flash memories

- In an array of cells, each cell stores a charge of certain level
- Absolute value of all charges varies due to leakage
- Relative order of charges may remain the same if leakage rate is roughly the same among cells
- Key idea: coding via permutations (rankings)

Recent work by: Barg, Bruck, Cassuto, Hagiwara, Jiang, Mazumdar, Schwartz, Yaakobi, Zhou,...

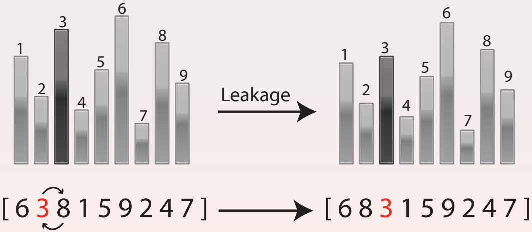
Rank Modulation

Rank modulation: record information via permutations of charge levels in arrays of cells
Example: permutation [6,3,8,1,5,9,2,4,7]



Rank Modulation

Errors occur due to different leakage rates in the form of swaps of two adjacent elements



Rank Modulation: Translocations

Translocation: cyclic shifts of elements.

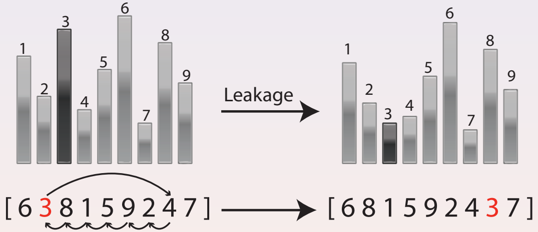


Figure : Translocation errors

Results

- Achievable coding rate for translocation errors
- Two families of codes for translocation errors
  - Interleaving codes with good Hamming distance
  - Interleaving codes with good Hamming distance and Translocation distance: asymptotically good codes
- Decoding algorithms

Permutations: A Formal Definition

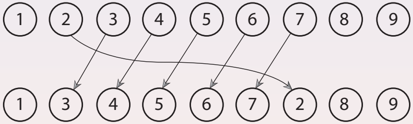
- Let $[n] = \{1, 2, \cdots, n\}$.
- A permutation is a bijection from $[n]$ to $[n]$.
- Set of all permutations of $[n]$: $\mathbb{S}_n$.
- Set of all permutations of P: $\mathbb{S}_P$, $P \subseteq [n]$.
- A permutation $\tau \in \mathbb{S}_n$ is a transposition if for $i, j \in [n]$,
  $\tau = (1, \cdots, i-1, \quad j, \quad i+1, \cdots, j-1, \quad i, \quad j+1, \cdots, n)$.

  If $j = i + 1$, then τ is called an adjacent transposition.

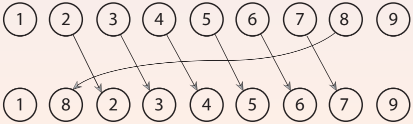
Translocations: A Formal Definition

A translocation $\phi(i, j)$ is a permutation obtained from identity by moving element i to position j and shifting everything in between by one position.

- Right translocation: if $i \leq j$, eg, $\phi(2, 7)$:



- Left translocation: if $i > j$, eg, $\phi(8, 2)$:



Distance Measures for Permutations

- Hamming distance: number of substitutions required to transform one permutation to another
- Transposition distance: number of transpositions required to transform one permutation to another
- Kendall τ distance: number of adjacent transpositions required to transform one permutation to another
- Translocation distance: number of translocations required to transform one permutation to another

Permutation Codes for ...

- Hamming distance: [Blake, Vinck, Kloeve, Colbourn,...]
- Transposition distance: Have not received any interest in the coding literature.
- Kendall τ distance: [Barg, Bruck, Cassuto, Hagiwara, Jiang, Mazumdar, Schwartz, Yaakobi, Zhou,...]
- Translocation distance:
  - Levenshtein's insertion/deletion codes:
    For permutations, 1 translocation = 1 deletion + 1 insertion
    Only efficient single-deletion-correcting codes known.
  - Beame and Blaise (Ulam distance): zero-rate regime

Bounds on the Size of Translocation Codes

Definition: For all $n, d \in \mathbb{Z}$, $A_\circ(n,d)$ denotes the largest number of permutations in $\mathbb{S}_n$ with pairwise translocation distance at least d.

Theorem: Let $\mathcal{C}_\circ := \lim \frac{\ln A_\circ(n,d)}{\ln n!}$ and $\delta = \lim \frac{d(n)}{n}$. Then

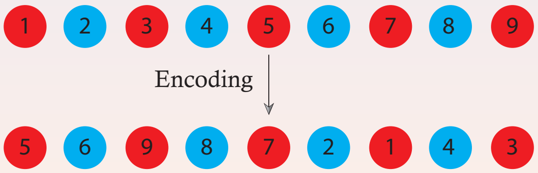
$$\mathcal{C}_\circ = 1 - \delta.$$

Theorem: For Hamming and Transposition codes, respectively,

$$\mathcal{C}_H = 1 - \delta, \qquad 1 - 2\delta \leq \mathcal{C}_T \leq 1 - \delta$$

Code Construction: Single Right-translocation Error

1. Partition $[n]$ into even and odd parts.
   Eg, $P_1 = \{1, 3, 5, 7, 9\}, P_2 = \{2, 4, 6, 8\}$.
2. Use codes with Hamming distance 2 over P_1 and P_2.
3. Interleave.



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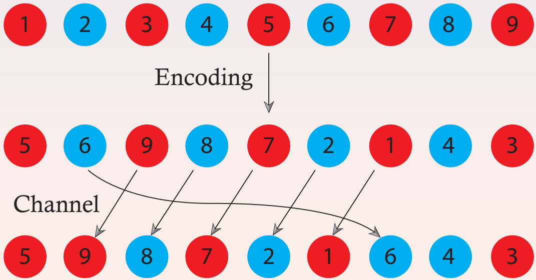

Code Construction: Single Translocation Error

1. Partition $[n]$ into three parts.
   Eg, $P_1 = \{1, 4, 7\}, P_2 = \{2, 5, 8\}, P_3 = \{3, 6, 9\}$.
2. Use codes with Hamming distance 2 over P_1, P_2, and P_3.
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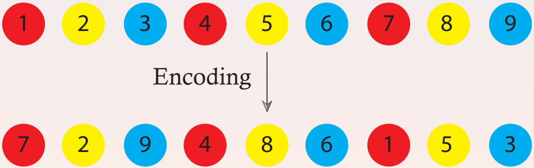

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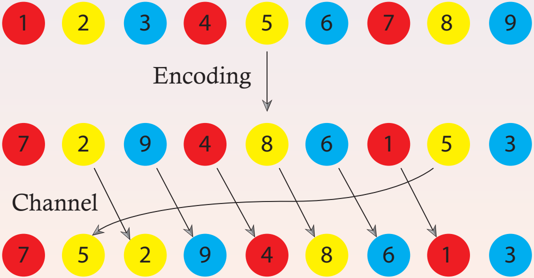
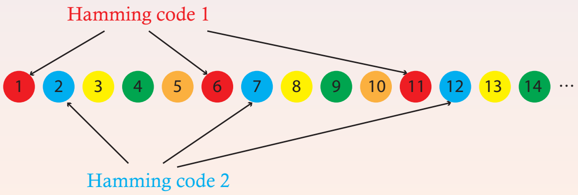


Interleaving Hamming Codes

- Partition $[n]$ into $2t+1$ classes P_i
- Choose C_i to be a permutation code over P_i with minimum Hamming distance at least $4t+1$.
- Construct code C by interleaving the codes C_i

Theorem: C corrects t errors.



Interleaving Hamming Codes

- Assuming best Hamming distance codes, cardinality equals

$$|C| = A_H\left(\left\lfloor \frac{n}{d} \right\rfloor, 2d\right)^d.$$

- For $d \sim n^{\beta}$,

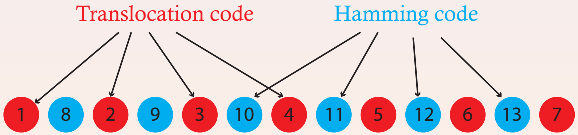
$$\lim \frac{\ln |C|}{\ln n!} = 1 - \beta.$$

- Decoding complexity: may be exponential in d.
- For constant d, decoding is polynomial (in n) and the rate approaches 1.

Interleaving Translocation and Hamming Codes

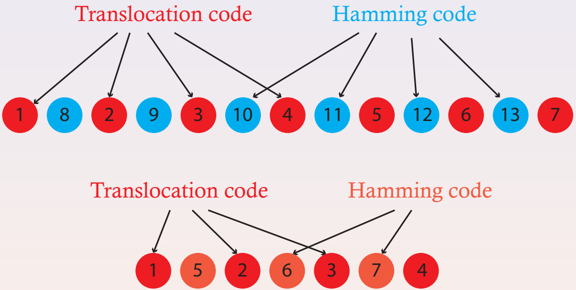
- Partition $[n]$ into P and Q of sizes p and $p-1$, with $n = 2p - 1$
- $C_1' \subseteq \mathbb{S}_P$ with minimum translocation distance d
- $C_1 \subseteq \mathbb{S}_Q$ with minimum Hamming distance $3d/2$

Theorem: The interleaved code has minimum translocation distance d.



Interleaving Translocation and Hamming Codes

The translocation code C_1' can be constructed in a similar manner.



Rate $\simeq 1 - 2^{-k}(1+k)$, k is the number of levels $= \left\lfloor \log \frac{2}{3\delta} \right\rfloor$

Decoding

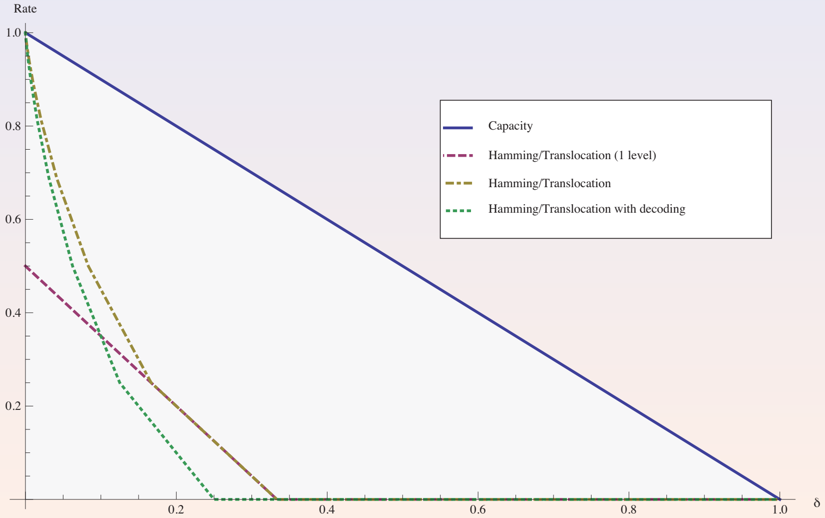
Decoding algorithm for interleaved codes (Translocation/Hamming)

- Translocation codes with min distance d
- Hamming codes used must have distance $2d$

Decoding algorithm is recursive: the inner-most components are decoded first.

The rate of code decodable with this algorithm is lower than what is implied by min distance since Hamming code must have distance $2d$ instead of $3d/2$.

Code Constructions: Asymptotic Rate Comparisons



Discussion

- Translocation distance is a lower bound for Hamming distance, Kendall distance, and transposition distance/2
- Translocation codes with large minimum distance are codes with large distances in above metrics
- In case of Hamming distance, no asymptotic rate loss