Lecture 16: Linear algebra with matrices

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Agenda

More about matrices
Gaussian elimination, solving linear systems
Finding the null space of a matrix

As they fly, airplanes can tilt or rotate along three axes (yaw, pitch, roll). Linear algebra can be used to solve a set of state equations for these and other dynamic parameters to maintain stable control of the aircraft.

Image credit: NASA/Glenn Research Center
More about matrices

Recall: a matrix is a finite, ordered sequence of numbers arranged in a rectangle with some number of rows and columns.

We already observed that these numbers influence the rank of the matrix, and the number of linearly independent vectors in the row/column spaces and corresponding null spaces of the matrix.

The shape and numbers in a matrix also yield other important properties that will be useful.
Matrix symmetry

If a matrix $A$ is square, and its off-diagonal elements satisfy $A(i,j) = A(j,i)$, we call the matrix \textit{symmetric}.

- Example: $\begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$

- Example: $\begin{pmatrix} 2 & 3 & 4 \\ 3 & -1 & -2 \\ 4 & -2 & 1 \end{pmatrix}$
Matrix symmetry

Suppose a matrix measures N columns and N rows. How many degrees of freedom does it have?

Now suppose it is also symmetric. How many degrees of freedom now?

For a general matrix A, with M rows and N columns, we call its transpose $A^T$, another matrix with N rows and M columns, such that $A^T(i,j) = A(j,i)$.

- For a symmetric matrix, $A = A^T$
Matrix symmetry

If we have a square matrix $A$, we can decompose it into *symmetric* and *antisymmetric* parts:

$$A_S = \frac{1}{2} (A + A^T), \quad A_{-S} = \frac{1}{2} (A - A^T)$$

- Verify $A_S$ is symmetric:

- What does antisymmetric mean?

Example: $A = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$
Matrix determinant

Given a square matrix, another important property is positive definiteness (or semidefiniteness). To understand this property, we should first introduce the matrix determinant:

For a 2x2 matrix, 
\[
\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc
\]

For an NxN matrix, we can compute determinants by taking all possible N-permutations (α, β, ..., ω) of the N columns (1, 2, ..., N) and computing the sum

\[
\begin{vmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{vmatrix} = \sum (-1)^{\text{flips}} a_{1\alpha} a_{2\beta} \cdots a_{N\omega}
\]

\[N! \text{ terms} – \text{yikes!}\]

- Flips for a permutation is how many interchanges we have to do to get from (1, 2, ..., N) to (α, β, ..., ω).
Matrix determinant

Cofactor formula provides more efficient way to compute determinants:

- For any value $a_{ij}$ in the square matrix $A$, its submatrix $M_{ij}$ is constructed by throwing out the $i$th row and $j$th column of $A$:

  $$
  A = \begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
  \end{pmatrix}, \quad \left( \begin{array}{c}
  \vdots \\
  \cdot \\
  a_{32} & \cdot
  \end{array} \right) \rightarrow M_{32} = \begin{pmatrix}
  a_{11} & a_{13} \\
  a_{21} & a_{23}
  \end{pmatrix}
  $$

- The cofactor $C_{ij}$ for value $a_{ij}$ is the determinant of $M_{ij}$, times $(-1)^{i+j}$.

- Then, the cofactor formula for the determinant of $A$ is the sum along any row of $A$ of the elements of that row, times the corresponding cofactors:

  $$
  |A| = \sum_{j=1}^{N} a_{ij}C_{ij} = \sum_{j=1}^{N} (-1)^{i+j}a_{ij}|M_{ij}|
  $$
Matrix determinant

Example: compute determinant of $A = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$
Matrix determinant

Cramer’s Rule describes how to use the determinant to solve a linear system of equations $Ax=b$:

$$x_j = \frac{|B_j|}{|A|} \text{ where } B_j \text{ is the matrix } A \text{ with the jth column replaced by vector } b$$

A *positive semidefinite* matrix $A$ satisfies the following for any size-compatible vector $x$:

$$x^T Ax \geq 0$$

In addition, this matrix is *positive definite* if and only if $x^T Ax > 0$ for all nonzero $x$.

- When a matrix $A$ is positive definite, the following determinants all must be $> 0$:

  $$|a_{11}|, |a_{11} a_{12} a_{13}|, |a_{11} a_{12} a_{13} a_{14}|, ..., |a_{11} \ldots a_{1N}|$$

- This is called *Sylvester’s criterion*
Positive definite matrices

In general, positive definite matrices have nice properties (especially when they are also symmetric). For example, when $A$ is positive definite, the quadratic function $x^T Ax - b^T x + c$ always has a unique minimum. This also implies that the linear system $A x = b$ has a unique solution.

It also means that the matrix transformation $A$ preserves some information about the size or length of a vector $x$, so that $y = Ax$ grows as $x$ grows.

Positive semidefinite matrices are less well-behaved, as the linear system $A x = b$ no longer has a unique solution, and the quadratic function does not have a unique minimum.
Gram Matrix

For a general matrix $A$ with $M$ rows and $N$ columns, we can construct its Gramian, or Gram matrix, $G$, by taking the matrix product $A^T A$.

- This Gram matrix is always symmetric and positive semidefinite.
- This Gram matrix is always square.
- The rank of $G$ is the same as the rank of $A$. If $G$ is full rank, $G$ is also positive definite.

Example: What is the Gram matrix of $A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$?
Solving linear systems

Let us start with solving a linear system with $N$ equations and $N$ unknowns:

\[
\begin{align*}
    a_{11}x_1 + \cdots + a_{1N}x_N &= b_1 \\
    \vdots \\
    a_{N1}x_1 + \cdots + a_{NN}x_N &= b_N
\end{align*}
\]

We already noted a method for solving this linear system by collecting the $a$’s into a matrix $A$, the $b$’s into a vector $b$, and using Cramer’s rule to solve for $x$ using determinants.

Two other powerful techniques:

1) Elimination – construct a matrix form of the linear system and solve through transformations

2) Matrix inversion – find the inverse $A^{-1}$ such that $A^{-1}A$ yields the identity matrix $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Gaussian elimination

Suppose we want to solve:

\[\begin{align*}
3x_1 + x_2 - 2x_3 &= 1 \\
-x_1 + 2x_2 + 3x_3 &= -2 \\
x_1 - x_2 + x_3 &= -1
\end{align*}\]

Collect coefficients into a matrix:

\[
\begin{pmatrix}
3 & 1 & -2 & 1 \\
-1 & 2 & 3 & -2 \\
1 & -1 & 1 & -1
\end{pmatrix}
\]

We will transform the left three columns of the matrix to become upper triangular:

\[
\begin{pmatrix}
3 & 1 & -2 & 1 \\
0 & 7/3 & 7/3 & -5/3 \\
0 & 0 & 3 & -16/7
\end{pmatrix}
\]

Back-substitution yields the solution \(x_3 = -16/21, x_2 = 1/21, x_1 = -4/21\)
Gaussian elimination

After forming the augmented matrix \((A \ b)\), here are the steps:

1. Starting from the first column, let the pivot be the first equation (row) with a nonzero value. If that’s not the first row, swap rows. From each of the equations (rows) below, multiply by the entry to eliminate and subtract entry-by-entry from that row below. The result should have eliminated (set to zero) all the values below the first row in the first column.

2. Proceed to the next column. Let the pivot be the next row with a nonzero value; if that’s not the next row, swap rows. From each of the equations (rows) below that one, multiply by the entry to eliminate and subtract entry-by-entry from that row below. The result should have eliminated the values below the pivot row for that column.

3. Proceed to the next column, and repeat, until we’ve reduced the \(A\) part of the augmented matrix to be upper triangular (zeros below all the pivot elements).
Worked example

Use Gaussian elimination to solve

\[ \begin{align*}
   x_1 + 2x_2 - x_3 &= 1 \\
   4x_1 + 9x_2 - 3x_3 &= 8 \\
   -2x_1 - 3x_2 + 7x_3 &= 10
\end{align*} \]
Your turn

Use Gaussian elimination to solve this one:

\[
\begin{align*}
\quad x_1 + x_2 + x_3 &= 6 \\
\quad x_1 + 2x_2 + 2x_3 &= 9 \\
\quad x_1 + 2x_2 + 3x_3 &= 10
\end{align*}
\]
Gaussian elimination to compute inverse

We can use Gaussian elimination to solve simultaneous linear systems (with the same A matrix):

\[ Ax_1 = b_1, Ax_2 = b_2, Ax_3 = b_3 \Rightarrow AX = B \]

Construct augmented matrix \((A \ B)\) and repeat pivot/elimination steps for all A columns.
Back-substitution will yield the solutions for each column of B separately.

We can also do this to compute the matrix inverse: perform Gaussian elimination of \((A \ I)\), where I is the identity matrix

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Then, if we continue with elimination to create zeros above the pivots, we get the reduced echelon form, which yields the Gauss-Jordan inverse solution.
Inverse computation example

Compute the matrix inverse of $A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix}$
Inverse computation – your turn

Compute the matrix inverse of \( A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \) using Gauss-Jordan elimination.
Inverse computation – your turn

Check your answer: multiple $A^{-1}$ by $A$, and $A$ by $A^{-1}$: (inverse should satisfy $A^{-1}A = AA^{-1} = I$)

$$A^{-1}A = \begin{pmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = $$

$$AA^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{pmatrix} = $$
Invertibility

The Gaussian elimination procedure hinges upon the assumption that we can find a pivot element for each column of $A$. If we cannot, we end up with a degenerate case of linearly dependent rows, such that the matrix $A$ cannot be inverted. This means $A$ is singular, does not have full rank, has a determinant equal to zero, and is not invertible. Thus, the Gaussian elimination procedure is a test for invertibility!

Example: try elimination on $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$. What is the determinant of $A$?
Review

Since elimination requires linearly independent rows (and columns), and invertibility is equivalent to being able to do elimination, and having an inverse also ensures having a nonzero determinant, and rank is connected to the number of linearly independent rows/columns, all these concepts are fundamentally connected:

- Linear dependence/independence
- Rank of a matrix
- Invertibility of a square matrix
- Nonzero determinant

It also turns out a positive definite matrix always will be invertible. (Converse is not true.)
The null space

We will spend some time working with the null space of matrices, so it makes sense to understand how to describe the null space of a matrix.

- The null space of any matrix, at a minimum, is a vector space containing the zero vector.
- More generally, if a matrix has reduced rank, it has a nontrivial null space, and the nullity is the number of linearly independent vectors needed to describe this null space.

To start, given a matrix, perform full row reduction on that matrix:

\[
\begin{pmatrix}
1 & -1 & 2 & 1 \\
-1 & 2 & 1 & 2 \\
2 & 1 & 2 & -2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & -21/11 \\
0 & 1 & 0 & -6/11 \\
0 & 0 & 1 & 13/11
\end{pmatrix}
\]
The null space

The remaining columns (non-pivoted) correspond to the free variables for the null space. In this example, just the last column is a free variable. So we write

\[
\begin{align*}
    x_1 - \frac{21}{11} x_4 &= 0 \\
    x_2 - \frac{6}{11} x_4 &= 0 \\
    x_3 + \frac{13}{11} x_4 &= 0
\end{align*}
\]

This allows us to express \( x_1, x_2, x_3 \) in terms of \( x_4 \), then write out the null space vector as:

\[
\begin{pmatrix}
    \frac{21}{11} x_4 \\
    \frac{6}{11} x_4 \\
    \frac{13}{11} x_4 \\
    -\frac{3}{11} x_4
\end{pmatrix}
\]
The null space

How about this rank-deficient matrix?

\[ A = \begin{pmatrix} 1 & 0 & -2 & 1 \\ -1 & 1 & -1 & 2 \\ -1 & 2 & -4 & 5 \end{pmatrix} \]

What is the rank of this matrix? (note: linearly dependent third column)

Compute reduced echelon form:

\[
\begin{pmatrix} 1 & 0 & -2 & 1 \\ -1 & 1 & -1 & 2 \\ -1 & 2 & -4 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]
The null space

So the null space has two free variables $x_3, x_4$:

$$x_1 - 2x_3 + x_4 = 0$$
$$x_2 - 3x_3 + 3x_4 = 0$$

So now the null space is a linear combination of two vectors:

$$\begin{pmatrix} 2x_3 - x_4 \\ 3x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -1 \\ -3 \\ 0 \\ 1 \end{pmatrix} x_4$$
Announcements

Due today: Lab 5 (two-part lab)

Next Thursday, March 28: Midterm #2

Next lecture: Error correction!