Agenda

Antialiasing filters and nonideal sampling
Reconstruction of analog signals
Sampling theorem
Orthogonality

Claude Shannon (1916-2001) was one of the co-discoverers of the Sampling Theorem, which governs the digital representation of analog signals. He also claimed to have designed a rocket-powered frisbee 😊

Image credit: Konrad Jacobs / Wikipedia
Antialiasing filter

In real world analog-to-digital conversion, we are not assured our signal is bandlimited.

This is where low pass filters come in. If we produce an analog low pass filter, we can attenuate, or suppress the analog frequencies above a chosen bandlimit, preventing aliasing in the low frequencies.

Note that analog filters are usually not as sharp or precise as their digital counterparts, but even a rough low pass analog filter can be enough to suppress aliasing.
Antialiasing filter and oversampling

If we sample faster, we can reduce the effects of quantization error on our digital signal spectrum.

*Oversampling* also has the advantage of permitting better antialiasing filtering:

- Use a rough analog antialiasing filter
- Sample at a faster rate \((f_s \times M)\)
- Use a sharp digital antialiasing filter (better attenuation)
- Discard excess samples after quantization

Real-world analog-to-digital converters frequently use this approach.
- Fast clocks, digital filters are relatively cheap.
Images and sampling

Working with images, we have two spatial dimensions (x and y).

Working with video, we also have a temporal dimension (t).

What are the relevant parameters related to sampling?
- In an image, the pixel size also reflects the spacing between samples. This is the image’s *resolution*.
- In a video, in addition to resolution, we also worry about *frame rate*.
Images and sampling

We observe that lower-resolution images do not necessarily contain all the information of a high-resolution image. For example:
Images and sampling

However, the sampling process in a spatial image is a bit different, in that our pixel averages (or integrates) all the light over its area. It is no longer an instantaneous sample.

This process of integration or averaging can be interpreted as a filter that distorts the signal’s spectrum. The distortion appears in frequency like a sinc function, a sine function that decays:

\[ x(t) = \begin{cases} 1, & |t| \leq W/2, \\ 0, & \text{otherwise} \end{cases} \quad \leftrightarrow \quad X(f) = \frac{\sin(\pi f W)}{\pi f} \]

Fourier transform of a box function is a sinc function
Pre-distortion

To undo the effects of this, our analog antialiasing filter can serve a second purpose:

- Instead of simply passing through the frequencies below the bandlimit, the filter can undo the distortion by filtering with an approximate inverse filter.
- Assuming the bandlimit is somewhere in the main lobe of the sinc function, the pre-distorting filter looks like this:
Videos and sampling

In addition to sampling in space, digital videos also sample in time.

The temporal sampling period is inversely proportional to the frame rate of the video.

While the video frames are acquired over very short time durations, how the video is displayed is not instantaneous.
- More about this when we discuss signal reconstruction next week.

Motion blur can be significant and is related to the frame rate and spatial resolution of the video.
Recall: frequency domain sampling

We notice the digital spectrum contains multiple copies of the analog spectrum:

These digital copies are centered at multiples of the sampling rate: 0, ±fs, ±2fs, ±3fs, ...

Let us first assume no aliasing...
Reconstruction as filtering

Idea: use frequency-selective filtering in analog domain to remove digital copies:

Question: why filter in analog domain? Why not digital?
Reconstruction as filtering

Can we process as digital? What would a digital filter look like? (Hint: it would also be periodic.)

Analog filter

Question: What should the “cutoff” frequency of this filter be?
Reconstruction example

Consider this signal with an analog spectrum reconstructed at $2^7 = 128$ Hz:

The signal is bandlimited. What is the bandwidth?
Reconstruction example

Let’s try increasing the digital signal’s sampling frequency to $2^8 = 256$ Hz:

We get a finer set of samples (right). What about the reconstruction?
Reconstruction example

Sketch the reconstruction from the 256 Hz samples:

Takeaway message: *oversampling* does not improve the fidelity of the reconstruction (it can help with sampling and quantization, though, as we discussed before).
Reconstruction example

What if the signal were sampled at $2^6 = 64$ Hz? This sampled signal has aliasing:

![Graph showing frequency and time plots with aliasing](image-url)
Reconstruction example

With aliasing in the frequency domain (not easily visible in time), the reconstruction will also be aliased:

Imperfect reconstruction results when the sampling rate is insufficiently high.
The Sampling Theorem

**Fundamental Tenet II:**

Perfect reconstruction of a bandlimited signal from a finite number of samples is possible by sampling above the Nyquist frequency.

This *perfect reconstruction* condition basically requires two things:

1) Signal must be *bandlimited* (or we can make it bandlimited with an antialiasing filter).

2) The sampling rate must be high enough so the periodic digital frequency copies don’t *overlap*.

Stated another way, the sampling rate $f_s$ must be greater than twice the bandwidth $W$. 
Suppose we were designing a digital system to process analog inputs (e.g., signals).

The sampling theorem tells us how fast we need to sample, not only to perfectly reconstruct the signal, but faithfully process the signal digitally:
Aliasing revisited

Consider the oversampled and undersampled cases:

- Oversampled: $W < \frac{fs}{2}$
- Undersampled (aliased): $W > \frac{fs}{2}$
Aliasing revisited

At $2W = f_s = \text{Nyquist frequency}$, we are critically sampled:

Note: the digital frequency copies exactly meet at $\pm W$
Sampling sinusoids

So, how fast do we need to sample a 1000 Hz sinusoid to reconstruct it?

Sampling theorem:

What if we knew it was specified with three parameters: amplitude, frequency, and phase?
Reconstructing sinusoids

Conversely, suppose we had a digital signal generator. What is the fastest analog signal we can reconstruct with this signal generator?

Image credit: user:Nmnogueira/Wikipedia
The reality of the sampling rate

Nonidealities demand we usually oversample relative to our analog signal’s bandwidth for accurate reconstruction:

- Imperfect / not exact antialiasing and lowpass reconstruction filters
- Quantization error mitigation

The *inconvenient* truth: neither analog nor digital filters are perfect!
Time and frequency representations

We observed that we can obtain a spectrum that portrays a signal’s frequency content.

We also remarked that the spectrum loses time information, so all the frequencies present in a signal (or an image) will be represented together.

This motivates the spectrogram:
- Multiply the signal by a series of “windows” centered at different time points and measure spectrum of each windowed signal individually.
Time and frequency uncertainty

However, this spectrogram represents a trade-off of information:

- Shorter time “windows” provide better time information for given frequency content.
- But, shorter time windows are less accurate for estimating that content.

This trade-off is in effect an uncertainty principle:

\[ \Delta_t \Delta_f \geq \frac{1}{2} \]

- As we use shorter time windows, \( \Delta_t \) will decrease, and \( \Delta_f \) will increase.
- This uncertainty principle is a fundamental limitation that is pervasive throughout physics. In fact, it is a major tenet of quantum physics that measuring a particle’s position will perturb its velocity, and vice versa. (Heisenberg Uncertainty Principle)
Uncertainty and windows

Using a *box* function in time to “window” a signal:
- Spread in time is limited to the width of the box.
- Spread in frequency is across the entire frequency range.

Using a spectral box function to measure frequency content:
- Spread in frequency is limited to the width of the box.
- Spread in time is across the entire signal time range, creating a $\text{sinc}(t)$ function in time.

Some functions like a *Gaussian* will provide limited spread in both time and frequency.

$$w(t) = e^{-\alpha(t-t_c)^2}$$

$\alpha > 0$ is a constant that controls the spread
$t_c$ is the center time of the window
Announcements

1) There are student(s) in our class who need a copy of the class notes, so the Student Disability Access Center is asking for a volunteer notetaker. Please consider doing this as a great service for one of your classmates. Please sign up via the SDAC Online Portal at: http://yukon.accessiblelearning.com/virginia/ApplicationNotetaker.aspx.

SDAC has some great prizes to raffle off to successful notetakers at the end of the semester. Prizes include gift certificates to local restaurants, shops, and entertainment such as Bodo's, Boylan Heights, and The Escape Room.

In addition to SDAC prizes, the volunteer notetaker will have my gratitude and collect bonus points towards the final exam (not that you’ll need them, because your notes will be so good!)
Announcements

2) We’ve created a *Slack* page for discussions among classmates. This site is not moderated or tracked by your TA’s or me, but we will check it periodically. Here is a link to sign up ([https://join.slack.com/t/uva19spmathof-lpm6043/signup](https://join.slack.com/t/uva19spmathof-lpm6043/signup))

- Note: anyone with a virginia.edu email can sign up.
- Anything posted on the site that violates UVA conduct guidelines (especially harassing or abusive language) will be reported and removed.
- We will not retain the site after the semester; the workspace will be deleted.
- Not joining or participating in this site will not affect your grade.
Announcements

3) Next time: Introduction to probability theory

Lab 3 today (ECE 2066)

Homework 4 out, due next Tuesday