As we will see, a bunch of signals (red, blue) can produce the same set of samples. This is due to aliasing.
Multiplication of signals

We have already discussed that adding a pair of signals together adds their spectra as well. The relationship for multiplying a pair of signals is more complicated, but equally as important.

To begin, consider the “simple” example of multiplying two cosine functions together:

\[ x_{12}(t) = x_1(t)x_2(t) = \cos(2\pi 3520t) \cos(2\pi 2093t) \]
Multiplication of signals

What happened? Let’s look at the spectrum:

1427 Hz

5613 Hz

3520 - 2093 = 1427 Hz

2093 + 3520 = 5613 Hz
Multiplication of signals

Multiplying two sinusoids together shifted the frequencies.

To see why, let’s review a trigonometric identity:

\[ 2 \cos(a) \cos(b) = \cos(a + b) + \cos(a - b) \]

Substituting \( a = 2\pi(3520)t \) and \( b = 2\pi(2093)t \), we get \( a+b = 2\pi(5613)t \) and \( a-b = 2\pi(1427)t \).
Recall: negative frequencies

It would have been okay to say the output has a frequency of \(2093-3520 = -1427\) Hz instead of \(+1427\) Hz.

Note that \(\cos(a) = \cos(-a)\) since cosine is an even function.

Thus, we could have shown the full spectrum, including these negative frequencies as well:
Multiplication of signals

What if one of them is not a sinusoid, but an arbitrary signal?

Multiplying by a sinusoid $\cos(2\pi f_1 t)$ creates shifted copies of the spectrum centered at $\pm f_1$.
Multiplication of signals

In general, multiplying two signals involves integrating over their spectra via frequency-domain convolution (don’t worry about the general case).

\[ x_{12}(t) = x_1(t)x_2(t) \iff X_{12}(f) = \int_{-\infty}^{\infty} X_1(f')X_2(f - f') \, df' \]

We are mainly interested in the setting where we are multiplying an arbitrary signal \( x(t) \) by a signal \( s(t) \) whose spectrum \( S(f) \) can be described as a set of “spikes” or delta functions.

\[ s(t) \iff S(f) = \sum_n \delta(f - f_n) \]

- In this case, the product spectrum looks like copies of the original signal spectrum \( X(f) \) centered around each spike in \( S(f) \).
- We will see that this process describes what happens when sampling an analog signal.
Modulation of signals

The process of multiplying a signal by a sinusoid is called *modulation*.

- This process is ubiquitous in signal processing and communication.
- It allows us to take a signal whose information is distributed among one set of frequencies and transform it into a signal whose information is distributed among a different set of frequencies, without losing anything important.
- In a communications medium, the frequency range available for transmission frequently does not match the frequency range of the information we want to transmit. Modulation solves this problem!
- The process of recovering a signal from its modulated version is called *demodulation*, and involves a very similar process.

- Let’s see an example...
Modulation example

Suppose we want to transmit human voice over a cellular signal.

- The human voice has an audible frequency range up to ~4 kHz.
- The LTE cellular signals range between 700 MHz – 2.7 GHz; for instance, 700 MHz is used by numerous cellular carriers including AT&T, Verizon, and T-Mobile.
- Our original audio signal $x(t)$ has a spectrum $X(f)$ nonzero between -4 kHz and +4 kHz.
- What frequency should our cosine modulator $\cos(2\pi f_1 t)$ be to transmit the signal in the 700 MHz band?

- What will our modulated spectrum look like?
Modulation example

Now suppose another caller wanted to transmit at the same time. One way to ensure their signals do not conflict is to assign different frequency bands to each caller. We will call this frequency division multiple access communication.

- Let’s modulate this caller to use 800 MHz band. What will their modulated spectrum look like?

- What about the combined spectrum?
Sampling

The main topic of this lecture is the process of sampling, how we get from analog (continuous-time) to digital (discrete-time) signals.

We introduced the idea of analog-to-digital conversion already, where we measure and quantize an analog signal at various instants in time.

\[ x_d[n] = Q(x_c(nT)) \]

Today, we will discuss what sampling actually looks like, as well as what actually happens to the information contained in an analog signal when we sample it.

- Short answer: we are frequency-limited based on how fast we can generate samples
Sampling on a chip

In the iPhone, the analog signal received through the microelectromechanical (MEMS) microphone sensor is converted to a digital signal via a piece of dedicated hardware called an application-specific integrated circuit (ASIC).

For instance, the iPhone 4 used a Knowles S1950 ASIC to handle the conversion:
We already discussed some types of analog-to-digital converters:

- Flash ADC
- Successive-approximation ADC

The sampling portion of the circuit typically contains at least these components:

- A clock generator (fixed or tunable)
- A latch circuit (triggered by the clock to measure the signal at a specific time)
- An analog low pass filter to pre-distort the signal and prevent aliasing (we’ll discuss more about this later)
Sampling as multiplication

By retaining only the signal values at particular instants in time, we are eliminating the signal content in between our samples. This is equivalent to multiplying the signal by a sequence of short rectangular pulses, T seconds apart:

Generally, these pulses are very short in duration, so we treat them as nearly instantaneous. The corresponding frequency spectrum of this sequence of pulses is a set of spikes, or delta functions, spaced out in frequency by $1/T$. We call this frequency the sampling rate, $f_s$.
Multiplication and sampling

Since we are multiplying the analog signal $x(t)$ by this sampling pulse $s(t)$, the spectrum $X_s(f)$ of the sampled signal will include reproductions of the analog spectrum $X_a(f)$ centered around each spike in the pulse spectrum $S(f)$:
Some intuition

One may wonder where this spike sequence spectrum comes from...

One approach to deriving it is by superimposing cosines of increasing frequency:

![Cosine at $f_s$](image)
Adding cosines together produces a sequence similar to our pulses:

![Graph showing cosines at fs, 2fs, 3fs](image-url)
Some intuition

Increasing the number of cosines results in sharper peaks and smaller variations throughout:

\[
s(t) \approx \frac{1}{N} \sum_{n=0}^{N} \cos(2\pi(nf_s)t)
\]
Sampling: analog and digital spectra

So we have analog and digital spectra. What do we notice about them?

- Periodicity of the spectrum
- Infiniteness of the spectrum

**Analog:** $X_a(f)$

**Sampled:** $X_s(f)$
Digital signal spectra

Technically, the shape of the spectrum is affected by quantization as well (we’ll ignore this).

Focusing just on the sampling step, a digital signal’s spectrum is periodic

- Period of spectrum (in Hz) = sampling frequency

- So, $X_s(f) = X_s(f + f_s)$

- How big should $f_s$ be? Or equivalently, how small should be $T = 1/f_s$?

- What assumption are we making about the analog signal?
Aliasing

*Aliasing* refers to the corruption of information in to this shifted copying process. It occurs when one spectrum copy overlaps with another.

- This overlap is like interference in physics, and it destroys our ability to resolve the overlapped information.
- In general, this overlap causes high-frequency content to be mistaken as low-frequency content.

- Have you watched a wheel spin really fast?
  - If it goes fast enough, it may even appear to spin backwards! Why?

We can understand aliasing in the frequency spectrum. What about in the time domain?
Aliasing in the time domain
Aliasing in the time domain

Sample slower ($f_s = 1.5$ kHz)
Aliasing in the time domain

Multiple signals would have the same samples at this rate:
Bandlimited signals

An analog (continuous-time) signal that has a finite range of nonzero frequency content is called *bandlimited*.

If our bandlimited signal has content up to positive frequency $W$, we call $W$ the *bandwidth* of the signal.

If our signal is bandlimited, we can choose $f_s$ large enough to avoid aliasing. How large should $f_s$ be?
Example

Suppose our signal is a sinusoid: $x(t) = 2\cos(2\pi(500)t + \pi/4)$.

What does the analog spectrum look like?

What is the bandwidth of this signal?

If we sample at 2000 Hz, what does the digital spectrum look like? What about 750 Hz?
Announcements
Next time: We will discuss the concept of signal reconstruction and how it relates to sampling

No lab today; extra office hours instead

Thursday – Midterm exam
Midterm exam #1

When: Thursday, February 7, during class (9:30 – 10:45 AM)

Where: In class (Olsson 120)

What: All the material up through last Thursday’s lecture

Policies:
• Bring one sheet (single sided 8½ x 11”) of notes, no photocopies allowed on the note sheet
• No books or other course materials are allowed
• Calculators are welcome but unnecessary (this is not a test on how to use a calculator)
• Make-up: please notify Prof. Weller ahead of time (if possible); being busy is not an excuse
Midterm Review