Quiz 14

Suppose X and Y are independent Laplace RVs with parameter λ: for all x and y,

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|},$$
$$f_Y(y) = \frac{\lambda}{2} e^{-\lambda|y|}.$$

Let $\begin{pmatrix} R \\ \Theta \end{pmatrix}$ be the polar coordinates of a point whose Cartesian coordinates are $\begin{pmatrix} X \\ Y \end{pmatrix}$. Find $f_{R,\Theta}$.

Solution

X and Y can be written as a function of R and Θ as

$$X = R\cos\Theta,$$
$$Y = R\sin\Theta.$$

Note that we are using the reverse mapping.

Consider pairs (r, θ) and (x, y) such that

$$x = r\cos\theta,$$
$$y = r\sin\theta.$$

So

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix},$$
$$|\det J| = r.$$

We have

$$f_{X,Y}(x, y) = \frac{1}{r} f_{R,\Theta}(r, \theta)$$

and thus

$$\begin{aligned} f_{R,\Theta}(r, \theta) &= r f_{X,Y}(x, y) \\ &= \frac{r\lambda^2}{4} e^{-\lambda(|x|+|y|)} \\ &= \frac{r\lambda^2}{4} e^{-\lambda r(|\cos\theta|+|\sin\theta|)}. \end{aligned}$$

It is clear that $f_{R,\Theta}$ depends on θ in the interval $\theta \in [-\pi, \pi]$ and thus $f_{R,\Theta}$ is not circularly symmetric. The figure below, also shows this fact.

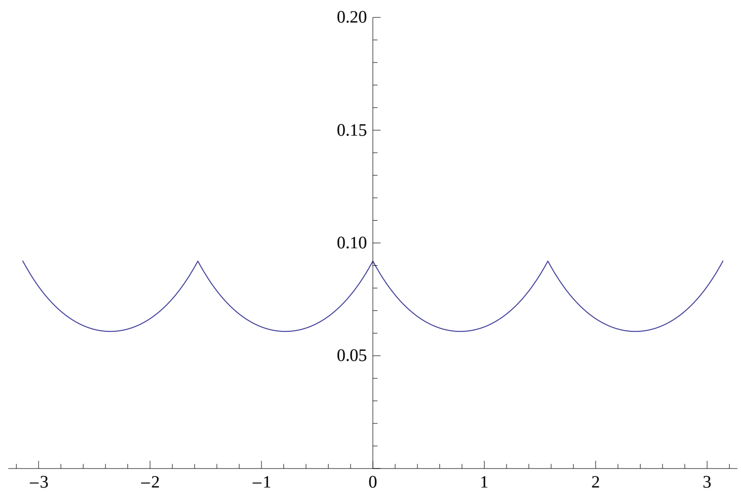

Figure 1: $f_{R,\Theta}(1,\theta)$ as a function of θ for $\lambda = 1$.