Quiz 8

Consider a Poisson process with rate λ. Define

- X: number of arrivals in the interval $[0, 2]$.
- Y: number of arrivals in the interval $[1, 3]$.
- Z: number of arrivals in the interval $[1, 2]$.

Find $P(Z = 2|X = 3, Y = 2)$.

Solution

Using the same notation as Example 3.5.5.b (and the same solution with minor changes!), we have

$$P(B_{302}) = \left(e^{-\lambda}\frac{\lambda^3}{3!}\right)\left(e^{-\lambda}\frac{\lambda^0}{0!}\right)\left(e^{-\lambda}\frac{\lambda^2}{2!}\right) = \frac{e^{-3\lambda}\lambda^5}{12},$$
$$P(B_{211}) = \left(e^{-\lambda}\frac{\lambda^2}{2!}\right)\left(e^{-\lambda}\frac{\lambda^1}{1!}\right)\left(e^{-\lambda}\frac{\lambda^1}{1!}\right) = \frac{e^{-3\lambda}\lambda^4}{2},$$
$$P(B_{120}) = \left(e^{-\lambda}\frac{\lambda^1}{1!}\right)\left(e^{-\lambda}\frac{\lambda^2}{2!}\right)\left(e^{-\lambda}\frac{\lambda^0}{0!}\right) = \frac{e^{-3\lambda}\lambda^3}{2},$$

and so

$$P(Z = 2, X = 3, Y = 2) = P(B_{120}) = \frac{e^{-3\lambda}}{12}\left(6\lambda^3\right),$$
$$P(X = 3, Y = 2) = P(B_{302}) + P(B_{211}) + P(B_{120}) = \frac{e^{-3\lambda}}{12}\left(\lambda^5 + 6\lambda^4 + 6\lambda^3\right).$$

Finally,

$$P(Z = 2|X = 3, Y = 2) = \frac{P(Z = 2, X = 3, Y = 2)}{P(X = 3, Y = 2)} = \frac{6}{\lambda^2 + 6\lambda + 6}.$$