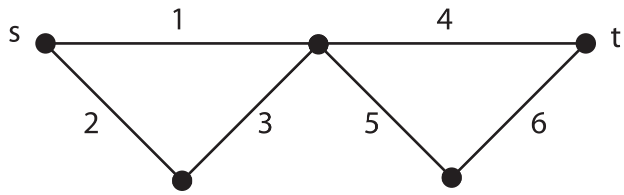
Quiz 7

Let X denote the capacity of the following network with source s and destination t. Links fail independently, each with probability p, and all have capacity 1. Determine the pmf of X.



Solution 1

The set of possible values: $\{0, 1, 2\}$. Let F_i denode the event that link i fails.

$$\begin{aligned}
p_X(0) &= P\left(\left(F_1\left(F_2 \cup F_3\right)\right) \cup \left(F_4\left(F_5 \cup F_6\right)\right)\right) \\
&= P\left(F_1F_2 \cup F_1F_3 \cup F_4F_5 \cup F_4F_6\right) \\
&= P\left(F_1F_2\right) + P\left(F_1F_3\right) + P\left(F_4F_5\right) + P\left(F_4F_6\right) \\
&\quad - P\left(F_1F_2F_3\right) - P\left(F_1F_2F_4F_5\right) - P\left(F_1F_2F_4F_6\right) - P\left(F_1F_3F_4F_5\right) - P\left(F_1F_3F_4F_6\right) - P\left(F_4F_5F_6\right) \\
&\quad + P\left(F_1F_2F_3F_4F_5\right) + P\left(F_1F_2F_3F_4F_6\right) + P\left(F_1F_2F_4F_5F_6\right) + P\left(F_1F_3F_4F_5F_6\right) \\
&\quad - P\left(F_1F_2F_3F_4F_5F_6\right) \\
&= 4p^2 - 2p^3 - 4p^4 + 4p^5 - p^6,
\end{aligned}$$

$$p_X(2) = P\left(F_1^cF_2^cF_3^cF_4^cF_5^cF_6^c\right) = (1-p)^6 .$$

$p_X(1)$ can be obtained as $1 - p_X(0) - p_X(2)$.

Solution 2

Let the middle point be denoted by u. Also let the capacity from s to u be denoted by Y and the capacity from u to t be denoted by Z. So we have $X = \min\{Y, Z\}$. Now,

$$\begin{aligned}
p_X(0) &= P\left(\{Y=0\} \cup \{Z=0\}\right) \\
&= P(Y=0) + P(Z=0) - P(Y=Z=0) \\
&= P(Y=0) + P(Z=0) - P(Y=0)\,P(Z=0) \\
&= P\left(F_1\left(F_2 \cup F_3\right)\right) + P\left(F_4\left(F_5 \cup F_6\right)\right) - P\left(F_1\left(F_2 \cup F_3\right)\right) P\left(F_4\left(F_5 \cup F_6\right)\right) \\
&= p\left(2p - p^2\right) + p\left(2p - p^2\right) - p^2\left(2p - p^2\right)^2 \\
&= 4p^2 - 2p^3 - 4p^4 + 4p^5 - p^6, \\
p_X(2) &= P(Y=2, Z=2) = P(Y=2)\,P(Z=2) = (1-p)^6 ,
\end{aligned}$$

and again $p_X(1) = 1 - p_X(0) - p_X(2)$.

Addendum to Solution 2

Just for fun, we also find $p_X(1)$ as follows

$$\begin{aligned}
p_X(1) &= P(Y=1, Z=1) + P(Y=1, Z=2) + P(Y=2, Z=1) \\
&= P(Y=1)\,P(Z=1) + P(Y=1)\,P(Z=2) + P(Y=2)\,P(Z=1) .
\end{aligned}$$

We have

$$\begin{aligned} P(Y=1) &= P\left(F_1^c\left(F_2 \cup F_3\right)\right) + P\left(F_1 F_2^c F_3^c\right) \\ &= (1-p)\left(2p - p^2\right) + p(1-p)^2 \\ &= p(1-p)(3-2p) \end{aligned}$$

and $P(Y=2) = (1-p)^3$. So

$$\begin{aligned} p_X(1) &= p^2(1-p)^2(3-2p)^2 + 2p(1-p)^4(3-2p) \\ &= 6p - 19p^2 + 22p^3 - 11p^4 + 2p^5. \end{aligned}$$

Note that since

$$(1-p)^6 = 1 - 6p + 15p^2 - 20p^3 + 15p^4 - 6p^5 + p^6,$$

we can verify that $p_X(0) + p_X(1) + p_X(2) = 1$.