ECE313		**Summary 2012**
	Problem Set 16	
Reading: Other stuff		Quiz Date: No Quiz

Note: It is very important that you solve the problems first and check the solutions afterwards.

Problem 1

Examples 4.8.2, 4.8.5, 4.9.1, 4.9.2, 4.11.4, 4.11.5.

Problem 2

Random variables X and Y have a uniform joint density on the square bounded by the following four corners: $(1,0),(0,1),(-1,0)$, and $(0,-1)$.

a) Calculate the marginal pdfs of X and Y. Are X and Y independent? Are they uncorrelated?

b) Let $Z = X + Y$ and $S = X - Y$. Are Z and S uncorrelated or independent or neither of the two?

c) Compute $E[X]$ and $\mathrm{Var}[X]$.

Solution

a) Let $\mathcal{R}$ be the square with corners $\{(1,0),(0,1),(1,0),(0,1)\}$. The area $|\mathcal{R}|$of $\mathcal{R}$ is $\sqrt{2}^2 = 2$. So

$$f_{X,Y}(x,y) = \begin{cases} 1/2, & (x,y) \in \mathcal{R} \\ 0, & \text{else.} \end{cases}$$

The marginal density of X is given by

$$\begin{aligned} f_X(x) &= \int_{y=-\infty}^{\infty} f_{X,Y}(x,y)dy && (1) \\ &= \begin{cases} \int_{y=x-1}^{1-x} \frac{1}{2}dy, & 0 \le x \le 1; \\ \int_{y=-x-1}^{1+x} \frac{1}{2}dy, & -1 \le x \le 0; \\ 0, & \text{else.} \end{cases} && (2) \\ &= \begin{cases} (1-x), & 0 \le x \le 1; \\ (1+x), & -1 \le x \le 0 \\ 0, & \text{else.} \end{cases} && (3) \\ &= \begin{cases} 1-|x|, & -1 \le x \le 1 \\ 0, & \text{else.} \end{cases} && (4) \end{aligned}$$

By symmetry, the marginal density of Y is given by

$$f_Y(y) = \begin{cases} 1-|y|, & -1 \le y \le 1 \\ 0, & \text{else.} \end{cases}$$

Independence: The support is not a product set, thus X and Y are *not* independent. Alternatively

$$\begin{aligned} f_X(x)f_Y(y) &= (1-|x|)(1-|y|) \\ &\neq \frac{1}{2} = f_{X,Y}(x,y) \end{aligned}$$

Uncorrelated: Since f_X (and f_Y) is symmetric about $x = 0$ (and $y = 0$), $E[X] = E[Y] = 0$. The covariance of X and Y is:

$$\begin{aligned} \text{Cov}(X,Y) &= E[XY] - E[X]E[Y] \\ &= 0 - 0 \end{aligned}$$

Thus, X and Y are uncorrelated. Alternatively,we can compute

$$\begin{aligned} E[XY] &= \int_x \int_y xy f_{X,Y}(x,y)dxdy \\ &= \int_{-1}^{1} \left(\int_{y=|x|-1}^{1-|x|} \frac{xy}{2} dy \right) dx \\ &= \int_{-1}^{1} \left(\frac{xy^2}{4} \Big|_{y=|x|-1}^{y=1-|x|} \right) dx \\ &= \int_{-1}^{1} \frac{x}{4} \underbrace{\left((|x|-1)^2 - (1-|x|)^2\right)}_{=0} dx \\ &= 0 \end{aligned}$$

And,

$$\begin{aligned} E[X] &= \int_x x f_X(x) dx \\ &= \int_{x=-1}^{0} x(1+x)dx + \int_{x=0}^{1} x(1-x)dx \\ &= -\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) \\ &= 0 \end{aligned}$$

Also, by symmetry $E[Y] = 0$. Hence, $E[XY] = E[X]E[Y] = 0$. So, X and Y are uncorrelated.

b) The transformation between (Z,S) and (X,Y) is given by

$$\begin{pmatrix} Z = X + Y \\ S = X - Y \end{pmatrix} \Leftrightarrow \begin{pmatrix} X = \frac{Z+S}{2} \\ Y = \frac{Z-S}{2} \end{pmatrix}$$

The transformed region $\mathcal{T}$ in the space of Z and S corresponding to region $\mathcal{R}$ in the space of X and Y can be represented as

$$\mathcal{T} = \{(z,s) : -1 \leq z \leq 1 \text{ and } -1 \leq s \leq 1\}$$

The joint density of Z and S is given by

$$\begin{aligned} f_{Z,S}(z,s) &= f_{X,Y}\left(\frac{z+s}{2}, \frac{z-s}{2}\right) \left| \begin{matrix} \frac{dx}{dz} & \frac{dx}{ds} \\ \frac{dy}{dz} & \frac{dy}{ds} \end{matrix} \right| \\ &= \frac{1}{2} \left| \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{matrix} \right|, \qquad (z,s) \in \mathcal{T} \\ &= \frac{1}{2} \times \frac{1}{2}, \qquad (z,s) \in \mathcal{T} \\ &= \frac{1}{4}, \qquad (z,s) \in \mathcal{T} \end{aligned}$$

Hence, Z and S are uniformly jointly distributed over the square given by $\{(z,s) : -1 \leq z \leq 1 \text{ and } -1 \leq s \leq 1\}$ of area 4.

Marginal pdf of Z and S:

$$\begin{aligned} f_Z(z) &= \int_{s=-1}^{1} f_{Z,S}(z,s)ds \\ &= \int_{s=-1}^{1} \frac{1}{4}ds \qquad -1 \le z \le 1 \\ &= \frac{1}{2}, \qquad -1 \le z \le 1 \end{aligned}$$

By symmetry, $f_S(s) = \frac{1}{2}, -1 \le s \le 1$.
Independence: Z and S are independent because,

$$\begin{aligned} f_Z(z)f_S(s) &= \frac{1}{2} \times \frac{1}{2}, \qquad (z,s) \in \mathcal{T} \\ &= \frac{1}{4} = f_{Z,S}(z,s) \end{aligned}$$

Uncorrelated: Independence implies uncorrelated. Hence, Z and S are uncorrelated.

c) From part (a),

$$E[X] = 0$$

Variance of X can be computed as

$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 \\ &= \int_{x=-1}^{1} x^2(1-|x|)dx - 0 \\ &= 2\int_{x=0}^{1} x^2(1-x)dx \\ &= 2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{6} \end{aligned}$$

Problem 3

Suppose X and Y are jointly Gaussian random variables with $E[X] = 2$, $E[Y] = 4$, $\mathrm{Var}(X) = 9$, $\mathrm{Var}(Y) = 25$, and $\rho = 0.2$. Let $W = X + 2Y + 3$.

a) Find $E[W]$ and $\mathrm{Var}(W)$.

b) Find the correlation and covariance of X and W.

Solution

a)

$$\begin{aligned} E[W] &= E[X] + 2E[Y] + 3 \\ &= 13 \end{aligned}$$

From the problem description, we know the following:

$$\begin{aligned}
E\left[X^2\right] &= \text{Var}\left(X\right) + E\left[X\right]^2 = 13 \\
E\left[Y^2\right] &= \text{Var}\left(Y\right) + E\left[Y\right]^2 = 41 \\
E\left[XY\right] &= 0.2\sqrt{\text{Var}\left(X\right)\text{Var}\left(Y\right)} + E\left[X\right]E\left[Y\right] \\
&= 11 \\
\text{Var}\left(W\right) &= E\left[W^2\right] - E\left[W\right]^2 \\
&= E\left[X^2\right] + 4E\left[XY\right] + 4E\left[Y^2\right] + 6E\left[X\right] + 12E\left[Y\right] + 9 - 169 \\
&= 121
\end{aligned}$$

b)

$$\begin{aligned}
\text{corr}\left(X, W\right) &= E\left[XW\right] \\
&= E\left[X^2 + 2XY + 3X\right] \\
&= 41 \\
\text{Cov}\left(X, W\right) &= \text{corr}\left(X, W\right) - E\left[X\right]E\left[W\right] \\
&= \text{corr}\left(X, W\right) - 26 \\
&= 15
\end{aligned}$$

Problem 4

If you drop a raw egg onto a concrete floor, what is the probability that you crack it?

Solution

Virtually zero; a concrete floor is very hard to crack.

Problem 5

This problem is concerned with minimum mean square error estimators.

a) Find the constant minimum mean square error estimator of the random variable $3X$, where X has mean $E[X] = 3$ and $\text{Var}\left(X\right) = 4$.

b) Find the linear minimum mean square error estimator of the random variable $2X$, where X has mean $E[X] = 3$ and $\text{Var}\left(X\right) = 4$, given an independent random variable Y with mean 2.

Solution

a)

$$\delta = E\left[3X\right] = 3E\left[X\right] = 9.$$

b)

$$\begin{aligned}
L^*(Y) &= \mu_{2X} + \sigma_{2X}\rho_{Y,(2X)}\frac{Y - \mu_Y}{\sigma_Y} \\
&= \mu_{2X} \\
&= 6.
\end{aligned}$$

Problem 6

Let $X \sim N\left(0, a^2\right)$ and $Y \sim N\left(0, b^2\right)$ and suppose X, Y are jointly Gaussian with correlation coefficient ρ. Define $Z = X + Y$. Is the pair (Z, X) jointly Gaussian? Find $\operatorname{Var}(Z)$ and $\operatorname{Cov}(Z, X)$.

Solution

Since

$$\begin{pmatrix} X \\ Z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix},$$

$\begin{pmatrix} X \\ Z \end{pmatrix}$ are jointly Gaussian.

$$\begin{aligned} \operatorname{Cov}(X, Z) &= \operatorname{Cov}(X, X+Y) = \operatorname{Cov}(X, X) + \operatorname{Cov}(X, Y) = a^2 + \rho ab, \\ \operatorname{Var}(Z) &= \operatorname{Cov}(X+Y, X+Y) = \operatorname{Var}(X) + 2\operatorname{Cov}(X, Y) + \operatorname{Var}(Y) = a^2 + b^2 + 2\rho ab. \end{aligned}$$