ECE313	**Summer 2012**
Problem Set 15	
Reading: Sum of jointly cont. RVs, examples of joint RVs, func. of joint RVs	Quiz Date: Tue, July 31

Note: It is very important that you solve the problems first and check the solutions afterwards.

Problem 1

Example 4.6.4 and Example 4.7.6.

Problem 2

Assume that X and Y are independent exponential random variables, both with parameter λ and support $(0, \infty)$. Find the joint pdf of the random variables $W = (X - Y)$ and $Z = \ln(X + Y)$.

Solution

The support for (X, Y) is $(0, \infty)^2$ and the support for (W, Z) is $(0, \infty)^2$. Over the support of (X, Y), the mapping is one-to-one, so

$$f_{W,Z}(w, z) = \frac{f_{X,Y}(x, y)}{|\det J|}$$

where $(w, z) = g(x, y) = (g_1(x, y), g_2(x, y)) = (x - y, \ln(x + y))$ and the J is the Jacobian of g.

$$J = \begin{bmatrix} 1 & -1 \\ \frac{1}{x+y} & \frac{1}{x+y} \end{bmatrix} \Rightarrow |\det J| = \frac{2}{x+y}.$$

We also need to find x, y in terms of w, z.

$$\begin{cases} w = x - y \\ z = \ln(x + y) \end{cases} \Rightarrow \begin{cases} w = x - y \\ e^z = x + y \end{cases} \Rightarrow \begin{cases} x = (e^z + w)/2 \\ y = (e^z - w)/2 \end{cases}$$

So for $w, z > 0$,

$$f_{W,Z}(w, z) = \frac{f_{X,Y}(x, y)}{|\det J|} = \frac{\lambda^2 e^{-\lambda(x+y)}}{\frac{2}{x+y}} = \frac{1}{2}\lambda^2 e^z e^{-\lambda e^z}$$

and $f_{W,Z}(w, z)$ equals 0 elsewhere.

Problem 3

Suppose X and Y are jointly continuous with joint pdf

$$f_{X,Y}(u, v) = \begin{cases} ve^{-(1+u)v}, & u, v \geq 0 \\ 0, & \text{else.} \end{cases}$$

a) Find the marginal pdfs, f_X and f_Y.

b) Find the conditional pdfs, $f_{Y|X}$ and $f_{X|Y}$: Be sure to indicate where these functions are well defined, and where they are zero, as well as giving the nonzero values.

c) $E[X|Y]$ is defined as follows. First define the function $g(v) = E[X|Y=v]$. Then define $E[X|Y]$ as $g(Y)$. Note that $E[X|Y]$ is a random variable.
Find $E[X|Y]$ and $E[Y|X]$.

d) Find the joint CDF, $F_{X,Y}(u,v)$

e) Are X and Y independent? Justify your answer.

Solution

a) For $u \geq 0$,

$$\begin{aligned} f_X(u) &= \int_0^\infty v \exp(-(1+u)v)\, dv \\ &= \frac{1}{(1+u)^2} \int_0^\infty te^{-t} dt \qquad \text{(we have let } t = (u+1)v \text{)} \\ &= \frac{1}{(1+u)^2} \left(e^{-t}(t+1)\right)_0^\infty = \frac{1}{(1+u)^2} \end{aligned}$$

and $f_X(u) = 0$ otherwise.
For $v \geq 0$,

$$\begin{aligned} f_Y(v) &= \int_{u=0}^\infty v \exp(-(1+u)v)\, du \\ &= [-\exp(-(1+u)v)]_{u=0}^\infty \\ &= \exp(-v) \end{aligned}$$

$f_Y(v) = 0$ otherwise.

b) If $u, v \geq 0$,

$$\begin{aligned} f_{Y|X}(v|u) &= \frac{f_{X,Y}(u,v)}{f_X(u)} \\ &= (1+u)^2 v \exp(-(1+u)v) \end{aligned}$$

If $u \geq 0, v < 0$, then $f_{Y|X}(v|u) = 0$. Otherwise $f_{Y|X}(v|u)$ is undefined.
If $u \geq 0, v \geq 0$

$$\begin{aligned} f_{X|Y}(u|v) &= \frac{f_{X,Y}(u,v)}{f_Y(v)} \\ &= v \exp(v) \exp(-(1+u)v) \\ &= v \exp(-uv) \end{aligned}$$

If $u < 0, v \geq 0$, then $f_{X|Y}(u|v) = 0$. Otherwise $f_{X|Y}(u|v)$ is undefined.

c) We find $E[X|Y=v]$. For $v > 0$,

$$\begin{aligned} E[X|Y=v] &= \int_0^\infty u f_{X|Y}(u|v) du \\ &= \int_{u=0}^\infty uv \exp(-uv)\, du \\ &= \frac{1}{v} \end{aligned}$$

Otherwise, $E[X|Y=v]$ is undefined. So $E[X|Y] = 1/Y$.
For $u \geq 0$,

$$\begin{aligned} E[Y|X=u] &= \int_0^\infty v f_{Y|X}(v|u) dv \\ &= \int_0^\infty v^2 (1+u)^2 \exp(-(1+u)v)\, dv \\ &= \frac{2}{1+u} \qquad u \geq 0 \end{aligned}$$

Otherwise, $E[Y|X=u]$ is undefined. So $E[Y|X] = 2/(1+X)$.

d) For $u, v \geq 0$,

$$\begin{aligned} F_{X,Y}(x,y) &= \int_{v=0}^{y} \int_{u=0}^{x} v \exp(-(1+u)v)\, du dv \\ &= \int_{v=0}^{y} \left[-exp(-(1+u)v)\right]_{u=0}^{x} dv \\ &= \int_{v=0}^{y} 1 - \exp(-(1+x)v)\, dv \\ &= \left[v + \frac{1}{1+x}\exp(-(1+x)v)\right]_{v=0}^{y} \\ &= y + \frac{1}{1+x}\left(\exp(-(1+x)y) - 1\right) \end{aligned}$$

Otherwise, $F_{X,Y}(x,y) = 0$.

e) Since $f_{X,Y} \neq f_X f_Y$, X and Y cannot be independent. The joint distribution cannot be factored implies the same result.