| **ECE313** | **Summer 2012** |
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| Problem Set 13 | |
| Reading: Joint CDF, Joint pmf, Joint pdf | Quiz Date: Tue, July 24 (maybe) |

**Note: It is very important that you solve the problems first and check the solutions afterwards.**

# Problem 1

All text examples in section 4.3.

# Problem 2

Consider the following function

$$F(u,v) = \begin{cases} 0, & u+v \leq 1 \\ 1, & u+v > 1. \end{cases}$$

Is this a valid joint CDF. Why or why not? Prove your answer and show your work.

# Solution

Suppose that $F$ is the CDF of $(U,V)$. Consider the rectangle $R$ with vertices $\{(0,0),(0,2),(2,0),(2,2)\}$. Then,

$$P\{(U,V) \in R\} = F(2,2) - F(2,0) - F(0,2) + F(0,0) = -1.$$

Negative probability implies that our assumption that $F$ is a CDF is wrong.

Furthermore, $F$ is not right continuous for any point $(u,v)$ such that $u+v=1$. Note that even if we make $F$ right-continuous by letting

$$F(u,v) = \begin{cases} 0, & u+v < 1 \\ 1, & u+v \geq 1, \end{cases}$$

it is still not valid because of the first reason.

# Problem 3

Suppose that two cards are drawn at random from a deck of 52 cards. Let $X$ be the number of queens obtained and let $Y$ be the number of spades obtained.

a) Find the joint probability mass function of $X$ and $Y$, the marginal probability mass function of $X$, and the marginal probability mass function of $Y$.

b) Find $P(X = Y)$.

c) Find $P(X \leq Y)$.

d) Find $P(X = 2|Y = 2)$.

## Solution

a) Some of probabilities are easy to find directly. Others may be more easily obtained by conditioning on $X$.

$$P(Y=i, X=0) = \begin{cases} \frac{36\cdot 35}{52\cdot 51}, & i=0, \\ \frac{2(12\cdot 36)}{52\cdot 51}, & i=1, \\ \frac{12\cdot 11}{52\cdot 51}, & i=2. \end{cases}$$

For $X=1$, we consider two cases: the queen is the queen of spades, or it is not.

$$P(Y=i, X=1, Q_{\spadesuit}) = \begin{cases} \frac{2(1\cdot 36)}{52\cdot 51}, & i=1, \\ \frac{2(1\cdot 12)}{52\cdot 51}, & i=2, \end{cases}$$

$$P\left(Y=i, X=1, Q_{\spadesuit}^c\right) = \begin{cases} \frac{2(3\cdot 36)}{52\cdot 51}, & i=0, \\ \frac{2(3\cdot 12)}{52\cdot 51}, & i=1, \end{cases}$$

where $Q_{\spadesuit}$ is the event that the queen of spades is chosen. Hence,

$$P(Y=i, X=1) = \begin{cases} \frac{2(3\cdot 36)}{52\cdot 51}, & i=0, \\ \frac{4\cdot 36}{52\cdot 51}, & i=1, \\ \frac{2(1\cdot 12)}{52\cdot 51}, & i=2. \end{cases}$$

Finally,

$$P(Y=i, X=2) = \begin{cases} \frac{3\cdot 2}{52\cdot 51}, & i=0, \\ \frac{2(1\cdot 3)}{52\cdot 51}, & i=1. \end{cases}$$

So we get

| | $X=0$ | $X=1$ | $X=2$ | Marginal of $Y$ |
|---|---|---|---|---|
| $Y=0$ | $\frac{210}{442}$ | $\frac{36}{442}$ | $\frac{1}{442}$ | $\frac{39\cdot 38}{52\cdot 51}=\frac{247}{442}$ |
| $Y=1$ | $\frac{144}{442}$ | $\frac{24}{442}$ | $\frac{1}{442}$ | $\frac{2(13\cdot 39)}{52\cdot 51}=\frac{169}{442}$ |
| $Y=2$ | $\frac{22}{442}$ | $\frac{4}{442}$ | $\frac{0}{442}$ | $\frac{13\cdot 12}{52\cdot 51}=\frac{26}{442}$ |
| Marginal of $X$ | $\frac{48\cdot 47}{52\cdot 51}=\frac{376}{442}$ | $\frac{2(4\cdot 48)}{52\cdot 51}=\frac{64}{442}$ | $\frac{4\cdot 3}{52\cdot 51}=\frac{2}{442}$ | |

Note that the marginals can be either found directly or by summing up rows and columns.

b)

$$P(X=Y) = \frac{210+24+0}{442} = \frac{234}{442}.$$

c)

$$P(X \leq Y) = \frac{210+144+22+24+4+0}{442} = \frac{404}{442}.$$

d)

$$P(X=2|Y=2) = \frac{P(X=Y=2)}{P(Y=2)} = \frac{\frac{0}{442}}{\frac{26}{442}} = 0.$$

## Problem 4

The jointly continuous random variables $X$ and $Y$ have joint pdf

$$f_{X,Y}(u,v) = \begin{cases} 1.5, & 0 \leq u < 1,\ 0 \leq v < 1,\ 0 \leq u+v < 1, \\ 0.5, & 0 \leq u < 1,\ 0 \leq v < 1,\ 1 \leq u+v < 2, \end{cases}$$

and zero elsewhere.

a) Find the marginal pdf of $Y$.

b) Find $P(X+Y \geq 3/2)$.

c) Find $P\left(X^2+Y^2 \leq 1\right)$.

## Solution

The support is the square with vertices $\{(0,0),(0,1),(1,0),(1,1)\}$. On the triangle with vertices $\{(0,0),(0,1),(1,0)\}$, the pdf is 1.5 and on the triangle with vertices $\{(0,1),(1,0),(1,1)\}$, it is 0.5. Sketching the pdf and marking the triangles is helpful for understanding the solution.

a) For $v \in [0,1]$, we have $f_Y(v) = \int_0^1 f_{X,Y}(u,v)\,du = \int_0^{1-v} \frac{3}{2}du + \int_{1-v}^1 \frac{1}{2}du = \frac{3(1-v)+1-(1-v)}{2} = \frac{3-2v}{2}$. For $v \notin [0,1]$, we have $f_Y(v) = 0$.

b) $P\left(X+Y \geq \frac{3}{2}\right) = \frac{1}{2}\times$ area of the triangle with vertices $\left\{\left(1,\frac{1}{2}\right),\left(\frac{1}{2},1\right),(1,1)\right\} = \frac{1}{16}$.

c) Let $A$ be the area of the triangle with vertices $\{(0,0),(0,1),(1,0)\}$ and let $O$ be the area of the unit circle. Then,

$$P\left(X^2+Y^2 \leq 1\right) = \frac{3}{2}A + \frac{1}{2}(O/4 - A) = \frac{3}{4} + \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right) = 0.89.$$