ECE313		**Summary 2012**
	Problem Set 12	
Reading: CLT, ML, Functions of RVs, Failure rate, Hypothesis testing		Quiz Date: Fri, July 20

Note: It is very important that you solve the problems first and check the solutions afterwards.

Problem 1

Suppose T has a failure rate function $h(t)$. Find the CDF of T. What condition must $h(t)$ satisfy?

Solution

Let the pdf and the CDF of T be f and F, respectively. Note that the support of T is the set of non-negative reals and so $F(0) = 0$. We have

$$\begin{aligned} h(s) = \frac{F'(s)}{1 - F(s)} &\Rightarrow -h(s) = \frac{d}{ds} \ln(1 - F(s)) \\ &\Rightarrow -\int_0^t h(s)\, ds = \ln(1 - F(t)) - \ln(1 - F(0)) = \ln(1 - F(t)) \\ &\Rightarrow F(t) = 1 - e^{-\int_0^t h(s) ds}. \end{aligned}$$

By definition, $h(t) \geq 0$. Furthermore since $F(t) = 1 - e^{-\int_0^t h(s) ds}$ and since $F(\infty) = 1$, it is required that $\int_0^\infty h(s)\, ds = \infty$.

Problem 2

Examples 3.6.9, 3.8.3, 3.8.6, 3.8.9, 3.9.1, 3.10.2, 3.10.3

Problem 3

A manufacturer of resistors has two factories and the resistors are guaranteed to have a resistance value within 2Ω of the nominal value.

- Under hypothesis 0, resistors are manufactured at factory 0 and have a variation around the nominal value that is a random variable X which is uniformly distributed in the interval $(-2, 2)$ and
- under hypothesis 1, resistors are manufactured at factory 1 and have a variation around the nominal value that is a random variable X with pdf given by $f_X(u) = \frac{1}{4}(2 - |u|)$ for $u \in (-2, 2)$ and zero elsewhere.

a) State the ML decision rule in terms of a threshold test on the observed value of $|X|$.

b) State the MAP decision rule in terms of a threshold test on the observed value of $|X|$.

c) For what range (if any) of values of 0, does the MAP decision rule always chooses hypothesis 0 (no matter what the observed value of the random variable is)?

d) Calculate the false alarm, missed detection and average error probabilities for the ML decision rule assuming $P(H_0) = \pi_0 = 1/3$.

e) Calculate the false alarm, missed detection and average error probabilities for the MAP decision rule assuming $P(H_0) = \pi_0 = 1/3$.

Solution

Let $Y = |X|$. Let the pdf of Y under H_0 be denoted by f_0 and under H_1 be denoted by H_1. Then, (why?)

$$f_0(y) = \frac{1}{2}, \qquad 0 \le y \le 2,$$
$$f_1(y) = 1 - \frac{y}{2}, \qquad 0 \le y \le 2.$$

and 0 otherwise. Plotting these pdfs may be helpful for the rest of the problem.

a) The likelihood ratio is

$$\Lambda(Y) = \frac{1 - Y/2}{1/2} = 2 - Y.$$

Then, for ML,

$$\text{dec. } H_1 \iff \Lambda(Y) > 1 \iff Y < 1$$

or equivalently,

$$\text{dec. } H_1 \iff |X| < 1$$

b) The decision rule for MAP is

$$\text{dec. } H_1 \iff \Lambda(Y) > \pi_0/\pi_1 \iff Y < 2 - \pi_0/\pi_1.$$

c) For MAP, to always decide H_0, we must have $0 > 2 - \pi_0/\pi_1$ which is satisfied iff

$$\frac{\pi_0}{\pi_1} > 2 \iff \pi_0 > \frac{2}{3}.$$

d) For ML, we have

$$p_{fa} = P(Y < 1 | H_0) = \frac{1}{2},$$
$$p_m = P(Y > 1 | H_1) = \frac{1}{4},$$
$$p_e = \frac{p_{fa} + 2p_m}{3} = \frac{1}{3}.$$

e) For MAP, we have

$$p_{fa} = P\left(Y < 2 - \frac{\pi_0}{\pi_1} | H_0\right) = \frac{1}{2}\left(2 - \frac{\pi_0}{\pi_1}\right) = \frac{3}{4},$$
$$p_m = P\left(Y > 2 - \frac{\pi_0}{\pi_1} | H_1\right) = P\left(Y > \frac{3}{2} | H_1\right) = \frac{1}{2}\left(\frac{1}{2} \cdot \frac{1}{4}\right) = \frac{1}{16},$$
$$p_e = \frac{p_{fa} + 2p_m}{3} = \frac{7}{24}.$$

The probability of error for MAP is slightly smaller than that of ML.