ECE313	**Summer 2012**
Problem Set 9	
Reading: Continuous RVs and Poisson Processes	Quiz Date: No Quiz, ME2

Note: It is very important that you solve the problems first and check the solutions afterwards.

Poisson Process is not part of the exam

Problem 1

(Uniform Quantization Errors) Suppose that you have recorded an audio signal from a microphone, and that you have normalized the audio signal so that its samples, X, are continuous random variables, uniformly distributed over the range $-1 \leq X \leq 1$. In order to convert an analog audio signal into a digital audio signal, the most common strategy is uniform quantization. The real-valued signal X is encoded on hard disk using a binary integer C, computed according to

$$C = \begin{cases} 0 & X < -1 \\ \lfloor 2^{B-1}(X+1) \rfloor & -1 \leq X < 1 \\ 2^B - 1 & X \geq 1 \end{cases} \tag{1}$$

where B is a positive integer-valued parameter called the bit rate (measured in bits per audio sample), and $c = \lfloor y \rfloor$ is defined to be the largest integer such that $c \leq y$. Notice that, by this definition, C is a discrete random variable defined over the range $C \in \{0, \dots, 2^B - 1\}$.

Equation (1) is an information-losing transformation: once the audio signal has been digitized, it is not possible to reconstruct the original signal without error. The best we can do, usually, is to approximate the original signal using the approximate value $\hat{X}$, defined as:

$$\hat{X} = 2^{-(B-1)}C - (1 - 2^{-B}) \tag{2}$$

The quantization error, Q, is defined to be the difference between the reconstructed and original signals, thus

$$Q \equiv \hat{X} - X \tag{3}$$

a) What is $E[X]$?

b) What is $E[X^2]$? This quantity is often called the "power" of the signal X.

c) Suppose $B = 2$ bits/sample.

   i) Sketch $f_X(u)$, the pdf of X. Shade the region under the portion of this pdf corresponding to the event "$C = 3$." Based entirely on your sketch, and without doing any explicit integrals, find $P\{C = 3\}$.

   ii) Sketch $f_X(u)$, the pdf of X. Shade the region under the portion of this pdf corresponding to the event "$|\hat{X}| = \frac{1}{4}$" (in words: the signal X is quantized using some codeword C, then reconstructed using a value $\hat{X}$ such that the absolute value of $\hat{X}$ is $\frac{1}{4}$). Based entirely on your sketch, and without doing any explicit integrals, find $P(|\hat{X}| = \frac{1}{4})$.

d) Now allow B to be a free parameter. Let Q be the continuous random variable $Q = \hat{X} - X$, and define its pdf to be $f_Q(v)$. Sketch $f_Q(v)$, and label both axes in terms of B.

e) The signal-to-noise ratio level (SNRL, in decibels) is defined to be

$$\text{SNRL} = 10\log_{10}\left(\frac{E[X^2]}{E[Q^2]}\right)$$

Find the SNRL in terms of B.

Solution

a)

$$E[X] = \int_{-1}^{1} \frac{u}{2} du = \left[\frac{u^2}{4}\right]_{-1}^{1} = 0$$

b)

$$E[X] = \int_{-1}^{1} \frac{u^2}{2} du = \left[\frac{u^3}{6}\right]_{-1}^{1} = \frac{1}{3}$$

c) For this part:

i) The sketch should be a rectangle of height 1/2, defined for the domain $-1 \leq u \leq 1$. The shaded portion should correspond to the subdomain $0.5 \leq u \leq 1$. The area of the shaded region is $P\{C = 3\} = \frac{1}{4}$.

ii) The event $|\hat{X}| = \frac{1}{4}$ occurs if $\hat{X} = \frac{1}{4}$ or if $\hat{X} = -\frac{1}{4}$. These two events correspond, respectively, to the events $-0.5 \leq X < 0$ and $0 \leq X < 0.5$. The shaded portion of $f_X(u)$ should therefore correspond to the subdomain $-0.5 \leq u \leq 0.5$. The area of the shaded region is $P(|\hat{X}| = \frac{1}{4}) = \frac{1}{2}$.

d) The sketch should be a rectangle of height 2^{B-1}, and distributed over the domain $-2^{-B} \leq v \leq 2^{-B}$.

e) Q is uniformly distributed between -2^B and 2^B, therefore $E[Q^2] = \frac{1}{3}4^{-B}$. The SNRL is therefore

$$\text{SNRL} = \log_M\left(\frac{1/3}{4^{-B}/3}\right) = 10B\log_{10}(4) \approx 6B$$

So each extra bit of quantization adds about 6 decibels of SNRL.

Problem 2

(The Laplacian: A Symmetrized Exponential) Samples of a speech or music signal are well modeled by a Laplacian pdf, defined as

$$f_X(u) = Ae^{-\lambda|u|}, \qquad -\infty < u < \infty$$

where $\lambda > 0$.

a) What is the value of A, in terms of λ?

b) Find the CDF, $F_X(c)$, by integrating $f_X(u)$.

c) What is $P\{|X| \leq c\}$, as a function of c?

d) Many interesting facts about X can be most quickly derived by using the law of total probability, combined with the following trick. Define an exponential random variable Y,

$$f_Y(v) = \begin{cases} \lambda e^{-\lambda v} & v \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then define the following experiment: First, Y is chosen at random, according to its pdf. Second, a fair coin is flipped. If the coin shows heads (event H), then X is set to $X = Y$. If the coin shows tails (event $T = H^c$), then X is set to $X = -Y$. Note that event H is independent of the value of random variable Y. By the law of total probability, this experiment implies the following formula; prove that this formula is satisfied by the CDF you found in part (b):

$$P(X \leq c) = P(H)P\{X \leq c|H\} + P(T)P\{X \leq c|T\} = P(H)P\{Y \leq c\} + P(T)P\{Y \geq -c\}$$

Be sure that your answer is correct regardless of whether $c \leq 0$ or $c \geq 0$.

e) Find $E[X]$ using the law of total probability.

f) Find $E[X^2]$ using the law of total probability.

g) Find $\mathrm{Var}(X)$.

Solution

a) $f_X(u)$ is symmetric with respect to $u = 0$, therefore

$$\int_{-\infty}^{\infty} Ae^{-\lambda u} du = 2\int_{0}^{\infty} Ae^{-\lambda u} du = \frac{2A}{\lambda}$$

and therefore $A = 0.5\lambda$.

b)

$$F_X(c) = \int_{-\infty}^{c} 0.5\lambda e^{-\lambda|u|} du = \begin{cases} 0.5e^{\lambda c} & c \leq 0 \\ 1 - 0.5e^{-\lambda c} & c \geq 0 \end{cases}$$

c) The absolute value of X is an exponential random variable. $P\{|X| \leq c\}$ is the CDF of this exponential random variable, therefore

$$P\{|X| \leq c\} = \begin{cases} 1 - e^{-\lambda c} & c \geq 0 \\ 0 & c < 0 \end{cases}$$

d) The component probabilities are as follows: $P(H^c) = P(H) = 0.5$, and

$$P\{X \leq c|T\} = P\{Y \geq -c\} = \begin{cases} e^{\lambda c} & c \leq 0 \\ 1 & c \geq 0 \end{cases}$$

$$P\{X \leq c|H\} = P\{Y \leq c\} = \begin{cases} 0 & c \leq 0 \\ 1 - e^{-\lambda c} & c \geq 0 \end{cases}$$

The CDF found in part (b) is easily shown to be half of $P(X \leq c|H^c)$ plus half of $P(X \leq c|H)$.

e) The pdf is symmetric with respect to $X = 0$, therefore it is intuitively obvious that $E[X] = 0$. This intuitively obvious fact can be proven as follows:

$$E[X] = 0.5E[X|H] + 0.5E[X|H^c] = 0.5\frac{1}{\lambda} + 0.5\frac{-1}{\lambda} = 0$$

f) The value of X^2 is the same as the value of Y^2. $E[X^2] = 0.5E[X^2|H] + 0.5E[X^2|H^c] = E[Y^2]$. Therefore $E[X^2] = E[Y^2] = \frac{2}{\lambda^2}$.

g) The random variable X is drawn from a larger variety of possible values than the random variable Y (X can be either $+Y$ or $-Y$), therefore intuition tells us that we should find $\mathrm{Var}(X) \geq \mathrm{Var}(Y)$. The book tells us that $\mathrm{Var}(Y) = 1/\lambda^2$. The definition of variance reveals that

$$\mathrm{Var}(X) = E[X^2] - E^2[X] = \frac{2}{\lambda^2}$$

Problem 3

(Wilma the Wombat) The video game "Wilma the Wombat" depicts the adventures of Wilma, a wombat in the Australian Outback, who is trying to cross a highway without getting hit by a car. The time, in minutes, between any two successive cars, T, is an exponential random variable,

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

It takes Wilma exactly six minutes to cross the road. If a car comes during that time, Wilma bounces back to her own side of the road, and must begin again.

a) What is the probability that Wilma successfully crosses the road on her first attempt?

b) Wilma decides to try the following strategy. Instead of starting across the road immediately, she waits for the first car. Immediately after the first car passes, she starts across the road. Using this strategy, what is the probability that she successfully crosses the road on her first attempt?

c) Being a talented hacker, you decide to change the rules of the game. You alter the code so that, instead of being hit by each car that passes, Wilma just ducks under the cars. A counter keeps track of the number of cars that pass, N, during the six minutes of Wilma's journey. What is the pmf, $p_N(i)$, of random variable N?

d) Continuing your modification of the game, you decide to add a magic gumdrop on the center line of the highway. Suppose it takes Wilma 3 minutes to reach the magic gumdrop, then she eats it instantaneously, then immediately begins her trek across the remainder of the road. During the remaining 3 minutes of her trek across the highway, if a car arrives, she eats the car, gaining extra points. Her total score for the game is as follows:

$$X = \begin{cases} 0 & \text{if any car arrives in } 0 \leq t < 3 \\ N & \text{if no cars arrive in } 0 \leq t < 3, \text{ and } N \text{ cars arrive in } 3 \leq t < 6 \end{cases}$$

What is the pmf, $p_X(k)$, of random variable X?

e) Suppose, now, that Wilma has been given a sack containing an infinite number of magic gumdrops, but that each one lasts for only one minute. In order to win the game, Wilma must eat three cars. In this new version of the game, Wilma is allowed to stay on the road as long as she likes, but in order to remain invulnerable (and continue eating cars), she must keep eating gumdrops at a rate of one per minute.

   i) Define S to be the time, in minutes, until arrival of the third car. What is the pdf, $f_S(t)$, of random variable S?

   ii) Define G to be the number of gumdrops that Wilma eats before arrival of the first car (assuming that she is eating gumdrops at a rate of one per minute). Find the pmf $p_G(g)$.

   iii) Define D to be the number of gumdrops that Wilma eats, from the time she begins the game until arrival of the third car (assuming that she is eating gumdrops at a rate of one per minute). Find the pmf $p_D(d)$.

Solution

a) The exponential random variable is memoryless, therefore the amount of time that we must wait until arrival of the first car is an exponential random variable, even if we have no idea when the previous car came through. Thus

$$P(\text{Wilma succeeds}) = P(T > 6) = 1 - F_T(6) = e^{-6\lambda}$$

b) The exponential random variable is memoryless, therefore the amount of time that we must wait until arrival of the next car is an exponential random variable, regardless of when we start. Thus Wilma's new strategy does not change her chance of success at all; it is still

$$P(\text{Wilma succeeds}) = P(T > 6) = 1 - F_T(6) = e^{-6\lambda}$$

c) Exponential inter-arrival times define a Poisson process. The number of arrivals in a Poisson process, in six minutes, is a Poisson random variable with parameter 6λ, thus

$$p_N(i) = \begin{cases} \frac{(6\lambda)^i e^{-6\lambda}}{i!} & i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

d) There are three cases to consider. First, $X \leq -1$: this never happens. Second, $X \geq 1$: this happens only if there are no cars in the first three minutes (probability: $e^{-3\lambda}$), followed by $X = k$ cars in the next three minutes (probability: $\frac{(3\lambda)^k e^{-3\lambda}}{k!}$); since non-overlapping periods of time are independent, we can find the total probability that $X = k$ by multiplying these two probabilities. Third, $X = 0$: there are two mutually exclusive ways in which this can happen, so we compute its probability by adding the probabilities of the mutually exclusive events. Either a car arrives in the first three minutes (probability $1 - e^{-3\lambda}$), or no cars arrive during any of the six minutes (probability $e^{-6\lambda}$). Thus:

$$p_X(k) = \begin{cases} 0 & k \leq -1 \\ 1 - e^{-3\lambda} + e^{-6\lambda} & k = 0 \\ \frac{(3\lambda)^k e^{-6\lambda}}{k!} & k \geq 1 \end{cases}$$

e) For this part:

i) S is an Erlang random variable with parameters λ and $r = 3$, therefore

$$f_X(t) = \begin{cases} \frac{\lambda^3 t^2 e^{-\lambda t}}{2!} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

ii) G is equal to the number of minutes during which no cars appear, plus one. Define p to be the probability that one or more cars arrive in any given minute. Then $p = 1 - e^{-\lambda}$. Since non-overlapping intervals of time are independent, G is a geometric random variable with parameter p:

$$p_G(g) = \begin{cases} p(1-p)^{g-1} = (1 - e^{-\lambda})e^{-\lambda(g-1)} & g \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

iii) This one doesn't simplify to a known random variable. There are several different ways to calculate the correct answer.

i. The easiest way to solve this problem is to note that

$$p_D(d) = P(N_d \leq 2) - P(N_{d+1} \leq 2)$$

so

$$\begin{aligned} p_D(d) &= P(N_d \leq 2) - P(N_{d+1} \leq 2) \\ &= e^{-\lambda d}\left(1 + \lambda d + \frac{(\lambda d)^2}{2}\right) - e^{-\lambda(d+1)}\left(1 + \lambda(d+1) + \frac{(\lambda(d+1))^2}{2}\right). \end{aligned}$$

ii. The event $\{D = d\}$ can be written in terms of two random variables, N_d and N_1', where N_d is the number of cars that pass in the first d minutes (a Poisson RV with parameter λd), and N_1' is the number of cars that pass in the $(d+1)^{st}$ minute (a Poisson RV with parameter λ, this has the same distribution as N_1, the number of cars passing in the first minute). Since these are non-overlapping periods of time, the corresponding random variables are independent, so we can write

$$\begin{aligned}
p_D(d) &= P(N_d = 0)P(N_1' \geq 3) + P(N_d = 1)P(N_1' \geq 2) + P(N_d = 2)P(N_1' \geq 1) \\
&= e^{-\lambda d}\left(1 - e^{-\lambda} - \lambda e^{-\lambda} - \frac{1}{2}\lambda^2 e^{-\lambda}\right) + \lambda d e^{-\lambda d}\left(1 - e^{-\lambda} - \lambda e^{-\lambda}\right) + \frac{1}{2}\lambda^2 d^2 e^{-\lambda d}\left(1 - e^{-\lambda}\right) \\
&= \frac{1}{2}e^{-\lambda d}\left(2 + 2\lambda d + \lambda^2 d^2\right) - \frac{1}{2}e^{-\lambda(d+1)}\left(2 + 2\lambda + \lambda^2 + 2\lambda d + 2\lambda^2 d + \lambda^2 d^2\right) \\
&= \frac{1}{2}e^{-\lambda d}\left(2 + 2\lambda d + \lambda^2 d^2\right) - \frac{1}{2}e^{-\lambda(d+1)}\left(2 + 2\lambda\left(d+1\right) + \lambda^2\left(d+1\right)^2\right).
\end{aligned}$$

iii. Finally, you could find $P(D = d) = P(d \leq S < d+1)$,

$$P(d \leq S < d+1) = \int_d^{d+1} \frac{\lambda^3 t^2 e^{-\lambda t}}{2} dt,$$

which can be solved using integration by parts to obtain

$$\begin{aligned}
p_D\left(d\right) &= \left(-\frac{1}{2}e^{-dt}\left(\lambda^2 t^2 + 2\lambda t + 2\right)\right)_{t=d}^{d+1} \\
&= \frac{1}{2}e^{-d\lambda}\left(\lambda^2 d^2 + 2\lambda d + 2\right) - \frac{1}{2}e^{-\lambda(d+1)}\left(\lambda^2\left(d+1\right)^2 + 2\lambda\left(d+1\right) + 2\right).
\end{aligned}$$