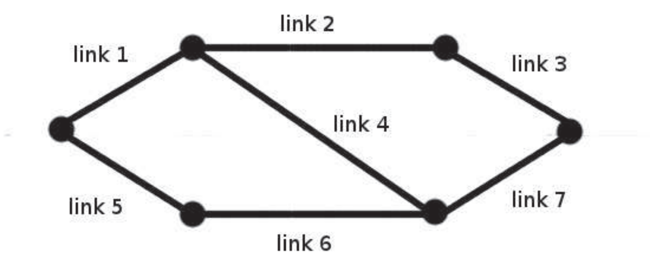
ECE313	**Summer 2012**
Problem Set 8	
Reading: Reliability, CDFs, and pdfs	Quiz Date: Fri, July 6

Note: It is very important that you solve the problems first and check the solutions afterwards.

Problem 1

(Capacity of a flow network) Consider the following $s-t$ flow network, where link $i \in \{2, \ldots, 6\}$ fails with probability p_i, while links 1 and 7 never fail. Let c_i be the capacity (see Section 2.12.3) of link i, then $c_1 = 30$, $c_2 = 10$, $c_3 = 15$, $c_4 = 20$, $c_5 = 20$, $c_6 = 15$, $c_7 = 15$.



a) What values can the capacity of this network take?

b) Find the distribution (pmf) of its capacity.

c) Find the numerical values of the pmf of the capacity if $p_i = \frac{1}{i}$ for all $i \in \{2, \ldots, 6\}$.

d) Using the numerical values from part (c), find the expected capacity of the network.

Solution

a) It can take the values $0, 10, 15, 25$.

b) We'll use C to denote the capacity, and L_i to denote the event that link i works. One can approach this problem by conditioning on whether link 4 is working or not. Therefore, using total probability, $p_C(k) = p_C(k|L_4)P(L_4) + p_C(k|L_4^c)P(L_4^c)$.
If link 4 is working, which occurs with probability $1 - p_4$, then we can always send 15 units through it from s to t, and if links 2 and 3 are working we can send an additional 10 units. Notice that whether links 5 and/or 6 work or not doesn't matter because we cannot send more units through link 7 than those we're already sending through link 4. Therefore, conditioned on link 4 working, the distribution of the capacity is

$$\begin{aligned} p_C(15|L_4) = & \quad P\{L_2^c \cup L_3^c\} = p_2 + p_3 - p_2 p_3 \\ p_C(25|L_4) = & \quad P\{(L_2^c \cup L_3^c)^c\} = P\{L_2 L_3\} = (1 - p_2)(1 - p_3). \end{aligned}$$

On the other had, if link 4 fails, which occurs with probability p_4, then the capacity depends on whether the top three links all work and/or the bottom three links all work. Therefore, conditioned on link 4 failing, the distribution of the capacity is

$$\begin{aligned}
p_C(0|L_4^C) = &\quad P\{(L_2^c \cup L_3^c)\,(L_5^c \cup L_6^c)\} = (p_2 + p_3 - p_2p_3)(p_5 + p_6 - p_5p_6) \\
p_C(10|L_4^C) = &\quad P\{L_2L_3\,(L_5^c \cup L_6^c)\} = (1-p_2)(1-p_3)(p_5 + p_6 - p_5p_6) \\
p_C(15|L_4^C) = &\quad P\{(L_2^c \cup L_3^c)\,L_5L_6\} = (p_2 + p_3 - p_2p_3)(1-p_5)(1-p_6) \\
p_C(25|L_4^C) = &\quad P\{L_2L_3L_5L_6\} = (1-p_2)(1-p_3)(1-p_5)(1-p_6).
\end{aligned}$$

The distribution of the capacity of the network can then be obtained from $p_C(k) = p_C(k|L_4)P(L_4) + p_C(k|L_4^c)P(L_4^c)$ as

$$\begin{aligned}
p_C(0) = &\quad (p_2 + p_3 - p_2p_3)(p_5 + p_6 - p_5p_6)p_4 \\
p_C(10) = &\quad (1-p_2)(1-p_3)(p_5 + p_6 - p_5p_6)p_4 \\
p_C(15) = &\quad (p_2 + p_3 - p_2p_3)(1-p_4) + (p_2 + p_3 - p_2p_3)(1-p_5)(1-p_6)p_4 \\
&= (p_2 + p_3 - p_2p_3)[1 - p_4(p_5 + p_6 - p_5p_6)] \\
p_C(25) = &\quad (1-p_2)(1-p_3)(1-p_4) + (1-p_2)(1-p_3)(1-p_5)(1-p_6)p_4 \\
&= (1-p_2)(1-p_3)[1 - p_4(p_5 + p_6 - p_5p_6)].
\end{aligned}$$

$p_C(k)$ is zero else.

c)

$$\begin{aligned}
p_C(0) = &\quad (\tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{2}\tfrac{1}{3})(\tfrac{1}{5} + \tfrac{1}{6} - \tfrac{1}{5}\tfrac{1}{6})\tfrac{1}{4} = \tfrac{2}{36} \\
p_C(10) = &\quad (1 - \tfrac{1}{2})(1 - \tfrac{1}{3})(\tfrac{1}{5} + \tfrac{1}{6} - \tfrac{1}{5}\tfrac{1}{6})\tfrac{1}{4} = \tfrac{1}{36} \\
p_C(15) = &\quad (\tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{2}\tfrac{1}{3})[1 - \tfrac{1}{4}(\tfrac{1}{5} + \tfrac{1}{6} - \tfrac{1}{5}\tfrac{1}{6})] = \tfrac{22}{36} \\
p_C(25) = &\quad (1 - \tfrac{1}{2})(1 - \tfrac{1}{3})[1 - \tfrac{1}{4}(\tfrac{1}{5} + \tfrac{1}{6} - \tfrac{1}{5}\tfrac{1}{6})] = \tfrac{11}{36}.
\end{aligned}$$

$p_C(k)$ is zero else.

d)

$$\begin{aligned}
E[C] = &\quad 0p_C(0) + 10p_C(10) + 15p_C(15) + 25p_C(25) \\
= &\quad 0\tfrac{2}{36} + 10\tfrac{1}{36} + 15\tfrac{22}{36} + 25\tfrac{11}{36} = \tfrac{615}{36} = \tfrac{205}{12} \approx 17.08.
\end{aligned}$$

Problem 2

(Analysis of a three-dimensional array code) This problem extends the analysis of Section 2.12.4 to a three dimensional array code. Consider a three dimensional array code with $7^3 = 343$ data bits and a total of $8^3 = 512$ bits so that the bits in every eight bit line (parallel to x axis, y axis, or z axis) has even parity.

a) What is the minimum number of bit errors in a nonzero undetected error pattern? Describe an error pattern with that many bit errors.

b) Suppose each bit is in error with probability $p = 0.001$, independently of the other bits. Using a union bound based on your answer to part (a), find an upper bound on the probability of undetected errors. (Hint: See www.ohrt.com/odds/binomial.php for a binomial coefficient calculator.)

c) How many undetected error patters are there with the minimum nonzero number of bit errors?

d) Show that the next smallest number of errors possible in an undetected error pattern is larger by at least four.

e) Using a union bound and the answers to (c) and (d), find a tighter upper bound on the probability of undetected error. (Hint: See the last part of Section 2.12.4 of the notes for a similar analysis.)

Solution

a) Undetected error patterns have the following constraint, implied by the analysis of the two dimensional array code. If there is at least one bit error in any two dimensional plane, there must be at least four bit errors. Also, not all the bit errors can be in a single two-dimensional plane–so there are at least two parallel two-dimensional planes with at least four bit errors each. So an undetected error pattern much have at least 8 bit errors. And 8 is possible–the locations of the bit errors would be a the corners of a three dimensional rectangular prism.

b) Let Y denote the total number of bit errors, which has the binomial distribution with parameters 512 and p. Then

$$P(\text{undetected error}) \leq P\{Y \geq 8\} \leq \binom{512}{8} p^8 = 1.11 \times 10^{-7}$$

c) Need to select two out of eight possible planes in each of the three dimensions, so there are $\binom{8}{2}^3 = 28^3 =$ 21952 such minimal error patterns.

d) If a nonzero undetected error pattern does not have 8 bit errors, then in at least one direction there has to be at least three parallel planes with at least one bit error each. Since a plane with at least one bit error must have at least four bit errors (to go undetected), there must be at least 12 bit errors. (Note: Twelve is possible, because there are undetectable error patterns for the two-dimensional array code with 6 bit errors, and repeating such pattern in two parallel planes in three dimensions gives an undetectable error pattern with 12 bits.)

e) $P(\text{undetected error}) \leq P\{Y = 8\} + P\{Y \geq 12\} \leq (21952)p^8 + \binom{512}{12} p^{12} = 2.19 \times 10^{-20} + 5.95 \times 10^{-13} \leq 6 \times 10^{-13}$

Problem 3

(CDFs) Consider the following function:

$$F(u) = \begin{cases} 0 & u < -5 \\ a & u = -5 \\ e^u & u \in (-5, -1) \\ b & u = -1 \\ \frac{1}{2} & u \in (-1, 1) \\ \frac{1}{8}u + \frac{3}{8} & u \in [1, 3) \\ 1 - ce^{-u} & u \geq 3 \end{cases}$$

a) Obtain the values (or range of values) for a, b, and c such that $F(u)$ is a valid CDF.

b) Sketch $F(u)$.
For the remainder of the problem supppose that $F(u)$ is the CDF of the random variable X, i.e. $F_X(u) = F(u)$.

c) Find $P\{X = -5\}$.

d) Find $P\{X = -2\}$.

e) Find $P\{X \leq -5\}$.

f) Find $P\{X \geq -1\}$.

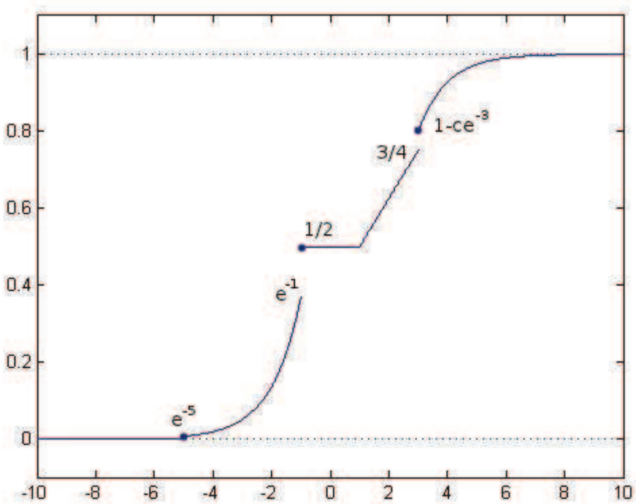
g) Find $P\{X = 0\}$.

h) Find $P\{X = 3\}$.

Solution

a) The values of the constants a and b are determined from the right-continuity of the CDF, i.e. $F(u) = F(u^+)$. Therefore, $a = F(-5) = F(-5^+) = e^{-5}$, and $b = F(-1) = F(-1^+) = 1/2$. The non-decreasing property of the CDF imposes an upper bound on the value of the constant c, i.e. $F(u^-) \leq F(u)$. Therefore $\frac{3}{4} = F(3^-) \leq F(3) = 1 - ce^{-3}$, so that $c \leq \frac{e^3}{4}$. The constant c is lower bounded due to the property that the CDF has to be less than or equal to one, i.e. $F(u) \leq 1$. Therefore, $c \geq 0$. In summary, $a = e^{-5}$, $b = 1/2$, and $0 \leq c \leq \frac{e^3}{4}$.

b) The plot is below.



c) $P\{X = -5\} = F_X(-5) - F_X(-5^-) = e^{-5} - 0 = e^{-5}$.

d) $P\{X = -2\} = F_X(-2) - F_X(-2^-) = e^{-2} - e^{-2} = 0$.

e) $P\{X \leq -5\} = F_X(-5) = e^{-5}$.

f) $P\{X \geq -1\} = 1 - P\{X < -1\} = 1 - F_X(-1^-) = 1 - e^{-1}$.

g) $P\{X = 0\} = F_X(0) - F_X(0^-) = \frac{1}{2} - \frac{1}{2} = 0$.

h) $P\{X = 3\} = F_X(3) - F_X(3^-) = 1 - ce^{-3} - \left(\frac{1}{8}3 + \frac{3}{8}\right) = \frac{1}{4} - ce^{-3}$.

Problem 4

(Power-Law Distributions) A power-law distribution is a pdf of the form

$$f_X(u) = \begin{cases} Au^{-b} & u \geq 1 \\ 0 & \text{else} \end{cases}$$

Few physical phenomena follow power-law distributions, but many social phenomena are distributed in this way. For example, if you choose an individual uniformly from the set of all humans on Earth, his or her annual income is approximately a power-law random variable. Discretized power-law distributions are also good models of the number of books sold by any given author, the number of times any given word is used in any given TV broadcast, and the number of living speakers of any given language. Power-law distributions are unusual in many respects. First, $f_X(u)$ is not a valid pdf for all possible values of the parameter b. Second, there are some values of b for which $f_X(u)$ is a valid pdf, but for which the expected value of X is unbounded (you can write "$E[X] = +\infty$"). Third, there are some values of b for which $E[X]$ is a well-behaved finite number, but for which $\mathrm{Var}(X) = +\infty$. The point of this problem is to explore some of these cases.

a) For each of the following values of the parameter b, either (1) find the value of A such that $f_X(u)$ is a valid pdf, or (2) prove that no such value exists.

   i) $b = 2$.

   ii) $b = 1.1$.

   iii) $b = .5$.

   iv) $b = 1$.

b) Find the set of parameter pairs (A, b) for which $f_X(u)$ is a valid pdf.

c) Suppose $b = 3$. What is $E[X]$?

d) Suppose $b = 2$. What is $E[X]$?

e) Suppose $b = 3$. What is $\mathrm{Var}[X]$?

Solution

a) For each case:

   i) $A = 1$.

   ii) $A = 0.1$

   iii) $\int_1^\infty Au^{-0.5}du = \frac{A}{0.5}[u^{0.5}]_1^\infty = \infty$. There is no A such that $\int_1^\infty Au^{-0.5}du = 1$.

   iv) $\int_1^\infty Au^{-1}du = [A\ln u]_1^\infty = \infty$. There is no A such that $\int_1^\infty Au^{-1}du = 1$.

b)

$$\int_1^\infty Au^{-b}du = \begin{cases} \frac{A}{b-1}, & b > 1 \\ \infty, & b \leq 1 \end{cases}$$

So $f_X(u)$ is a valid pdf if and only if $b > 1$, and $A = b - 1$.

c)

$$E[X] = \int_1^\infty 2u^{-2}du = 2$$

d)

$$E[X] = \int_1^\infty u^{-1}du = \infty$$

e)

$$E[X^2] = \int_1^\infty 2u^{-1}du = \infty$$

$$\mathrm{Var}(X) = E[X^2] - E^2[X] = \infty - 4 = \infty$$