| **ECE313** | **Summer 2012** |
|---|---|
| Problem Set 7 | |
| Reading: Cond. Prob., Law of total prob., Hypothesis testinng | Quiz Date: Tuesday, July 3 |

**Note: It is very important that you solve the problems first and check the solutions afterwards.**

# Problem 1

In the game of Risk, an *attack* is simulated by two players rolling dice. Player 1 rolls three red dice, and Player 2 rolls two white dice. Define the random variables $R_1$ and $R_2$ to be the largest and second largest numbers showing on the the red dice. Define $W_1$ and $W_2$ to be the higher and lower numbers shown on the white dice.

Points (which represent armies) are deducted from the players as follows:

| outcome | player 1 | player 2 |
|---|---|---|
| $R_1 > W_1$ and $R_2 > W_2$ | 0 | -2 |
| $R_1 > W_1$ and $R_2 \leq W_2$ | -1 | -1 |
| $R_1 \leq W_1$ and $R_2 > W_2$ | -1 | -1 |
| $R_1 \leq W_1$ and $R_2 \leq W_2$ | -2 | 0 |

Define the random variables $N_1$ and $N_2$ to be the number of points deducted from Player 1 and Player 2, respectively.

a) Determine the pmf for $R_2$.

b) Compute the pmf for $N_1$.

c) Compute the pmf for $N_2$.

d) Determine the expected number of points deducted from each player at the end of three consecutive attacks.

e) Suppose Player 2 has three armies. What is the expected number of attacks required to reduce his number of armies to zero (i.e., what is the expected number of attacks that will be required to penalize Player 2 -3 points)?

# Problem 2

A baseball pitcher's repertoire is limited to fastballs (event $F$), curve balls (event $C$) or sliders (event $S$). It is known that $P(C) = 2P(F)$. The event $H$, that the batter hits the ball has the conditional probabilities $P(H|F) = 2/5$, $P(H|C) = 1/4$, and $P(H|S) = 1/6$. Furthermore, based on this pitcher's past performance, it is estimated that $P(H) = 1/4$.

a) Compute the probabilities for the three pitches: $P(F)$, $P(S)$, and $P(C)$.

b) A fan sitting in the bleachers sees the batter getting a hit (i.e., the fan observes that the event $H$ occurs), but his seat is too far away to be able to tell which pitch was thrown. Compute the conditional probabilities for the three pitches: $P(F|H)$, $P(C|H)$, and $P(S|H)$.

## Problem 3

The dice game of craps begins with the player (called the shooter) rolling two fair dice. If the sum is 2, 3, or 12, the shooter loses the game. If the sum is a 7 or 11, the shooter wins the game. If the sum is any of 4, 5, 6, 8, 9, 10, then the shooter has neither won nor lost (as yet). The number rolled is called the shooter's point, and what happens next is described below in part (b).

a) What is the probability that the shooter wins the game on the first roll? What is the probability that the shooter loses the game on the first roll? What is the probability that the shooter's point is $i, i \in \{4, 5, 6, 7, 8, 10\}$?

b) Suppose that the shooter's point is $i$ for $i \in \{4, 5, 6, 7, 8, 10\}$. The shooter rolls the dice again. If the result is $i$, the shooter is said to have *made the point* and wins the game. If the result is 7, the shooter loses the game (craps out). If the result is anything else, the shooter rolls the dice again. This continues until the shooter either makes the point or craps out. For each $i \in \{4, 5, 6, 7, 8, 10\}$, compute the probability that the shooter wins the game. Note that these are conditional probabilities of winning given that the shooter's point is $i$.

c) Conditioned on the shooter's point being $i$, what is the expected number of dice rolls till the game ends? (Note: one dice roll = rolling two dice simultaneously). What is the expected number of dice rolls in a game of craps? What is the (unconditional) probability of winning at craps?

d) If the shooter's point is 8, then side-bets are offered at 10 to 1 odds that the shooter will make the point the hard way by rolling $(4, 4)$. Is this a fair bet? (Remember that 10 to 1 odds means if you bet a dollar, you will either lose the dollar, or you will win ten dollars (and will also get your original dollar back, of course!)).

## Problem 4

A transmitter can send one of three signals, $s_1, s_2, s_3$. The actual transmission is corrupted by noise, causing the received signal to be changed according to the following table of conditional probabilities.

| | | Receive $j$ | | |
|---|---|---|---|---|
| | | $s_1$ | $s_2$ | $s_3$ |
| Send $i$ | $s_1$ | .8 | .1 | .1 |
| | $s_2$ | .05 | .9 | .05 |
| | $s_3$ | .02 | .08 | .9 |

In this table, the entry of row $i$ and column $j$ corresponds to the conditional probability $P(s_j \text{ is received } | s_i \text{ is sent })$. Assume that the three signals are transmitted with equal probability.

a) Compute the probabilities that $s_1, s_2, s_3$ are received.

b) Compute the conditional probabilities $P(s_i \text{ is sent } | s_j \text{ is received })$ for $i, j = 1, 2, 3$.

## Problem 5

If $H_0$ is the true hypothesis, the random variable $X$ takes on values 0, 1, 2, and 3 with probabilities 0.1, 0.2, 0.3, and 0.4 respectively. If $H_1$ is the true hypothesis, the random variable $X$ takes on values 0, 1, 2, and 3 with probabilities 0.4, 0.3, 0.2, and 0.1 respectively.

a) Write down the likelihood matrix $L$ and indicate the maximum-likelihood decision rule by shading the appropriate entries in $L$. What is the false-alarm probability $P_{FA}$ and what is the missed-detection probability $P_{MD}$ for the maximum-likelihood decision rule?

b) Suppose that the hypotheses have *a priori* probabilities $\pi_0 = 0.7$ and $\pi_1 = 0.3$. Use the law of total probability to find the average error probability of the maximum-likelihood decision rule that you found in part (a).

c) Use the a priori probabilities given in part (b) to find the joint probability matrix $J$ and indicate on it the Bayesian decision rule, which is also known as the maximum a posteriori probability (MAP) decision rule. What is the average error probability of the Bayesian decision rule? Is it smaller or larger than the average error probability of the maximum- likelihood decision rule?

# Solution

a)

| $k$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X=k\|H_1)$ | $\underline{.4}$ | $\underline{.3}$ | .2 | .1 |
| $P(X=k\|H_0)$ | .1 | .2 | $\underline{.3}$ | $\underline{.4}$ |

Let $E$ denote the event that the decisions is wrong. For $P_{FA}$, look at the row corresponding to $P(\cdot|H_0)$. We have

$$P_{FA} = P(E|H_0) = .1 + .2 = .3.$$

For $P_M$, look at the row corresponding to $P(\cdot|H_1)$. We have

$$P_M = P(E|H_1) = .1 + .2 = .3.$$

b) We have

$$\begin{aligned} P(E) &= P(H_0)P(E|H_0) + P(H_1)P(E|H_1) \\ &= \pi_0 P_{FA} + \pi_1 P_M \\ &= 0.3. \end{aligned}$$

c)

| $k$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X=k,H_1) = \pi_1 P(X=k\|H_1)$ | $\underline{.12}$ | .09 | .06 | .03 |
| $P(X=k,H_0) = \pi_0 P(X=k\|H_0)$ | .07 | $\underline{.14}$ | $\underline{.21}$ | $\underline{.28}$ |

$$\begin{aligned} P_{FA} &= .1, \\ P_M &= .1 + .2 + .3 = .6, \\ P(E) &= \text{sum of the entries that are not underlined} \\ &= .07 + .09 + .06 + .03 \\ &= .25. \end{aligned}$$

Probability of error for MAP is smaller and that is exactly what we would expect.

# Problem 6

A transmitter chooses one of two routes (Route 0 or Route 1) and repeatedly transmits a packet over the chosen route until the packet is received without error (that is, without CRC checksum failure) at the receiver. Suppose an error-free feedback link allows the transmitter to know when the packet is transmitted without error. Let $X$ denote the number of times the packet is transmitted over the chosen route including the final error-free transmission. Assuming that the successive transmissions are independent trials of an experiment, the two hypotheses are

- $H_1$: Route 1 is used for packet transmission: $X \sim \text{Geo}(p_1)$

- $H_0$: Route 0 is used for packet transmission: $X \sim \text{Geo}(p_0)$

where $0 < p_1 < p_0 < 1$ are the probabilities of error-free transmission over the two routes.

a) State the maximum-likelihood decision rule as to which route was used as a threshold test on the observed value of $X$.

b) Suppose the transmitter chooses Route 0 and Route 1 with probabilities $\pi_0$ and $\pi_1 = 1 - \pi_0$ respectively. Assume that $0 < \pi_0 < 1$.
For what values of $\pi_0$ (if any) does the MAP decision rule always choose hypothesis $H_1$ regardless of the value of the observation $X$?
For what values of $\pi_0$ (if any) does the MAP decision rule always choose hypothesis $H_0$ regardless of the value of the observation $X$?

## Solution

a) First, find the likelihood ratio:

$$\begin{aligned}\Lambda(k) &= \frac{P(X = k|H_1)}{P(X = k|H_0)} \\ &= \frac{p_1 (1-p_1)^{k-1}}{p_0 (1-p_0)^{k-1}}.\end{aligned}$$

Suppose that ties are broken in favor of $H_0$. The ML rule is

$$\begin{aligned}\text{decide } H_1 &\iff \Lambda(k) > 1 \\ &\iff \log \Lambda(k) > 0 \\ &\iff \log\left(\frac{p_1}{p_0}\right) + (k-1)\log\left(\frac{1-p_1}{1-p_0}\right) > 0 \\ &\iff (k-1)\log\left(\frac{1-p_1}{1-p_0}\right) > \log\left(\frac{p_0}{p_1}\right) \\ &\overset{(\dagger)}{\iff} k - 1 > \frac{\log(p_0/p_1)}{\log\left(\frac{1-p_1}{1-p_0}\right)} \\ &\iff k > \frac{\log(p_0/p_1)}{\log\left(\frac{1-p_1}{1-p_0}\right)} + 1,\end{aligned}$$

where $(\dagger)$ follows since $0 < p_1 < p_0 < 1$ and thus $\log\left(\frac{1-p_1}{1-p_0}\right) > 0$.

b) Define $\tau = \frac{\pi_0}{\pi_1}$. Similar to the previous part, we have

$$\begin{aligned}
\text{decide } H_1 &\iff \Lambda(k) > \tau \\
&\overset{(\dagger)}{\iff} \log \Lambda(k) > \log \tau \\
&\iff \log\left(\frac{p_1}{p_0}\right) + (k-1)\log\left(\frac{1-p_1}{1-p_0}\right) > \log \tau \\
&\iff (k-1)\log\left(\frac{1-p_1}{1-p_0}\right) > \log\left(\frac{\tau p_0}{p_1}\right) \\
&\iff k-1 > \frac{\log(\tau p_0/p_1)}{\log\left(\frac{1-p_1}{1-p_0}\right)} \\
&\iff k > \frac{\log(\tau p_0/p_1)}{\log\left(\frac{1-p_1}{1-p_0}\right)} + 1, \qquad (1)
\end{aligned}$$

where $(\dagger)$ follows from the fact that $\tau > 0$ and its logarithm is defined. If $\frac{\log(\tau p_0/p_1)}{\log\left(\frac{1-p_1}{1-p_0}\right)} + 1 < 1$, we always decide in favor of $H_1$. Thus we must have

$$\log\left(\frac{\tau p_0}{p_1}\right) < 0 \iff \frac{\tau p_0}{p_1} < 1 \iff \frac{\pi_0}{1-\pi_0} < \frac{p_1}{p_0} \iff \pi_0 < \frac{p_1/p_0}{1+p_1/p_0} \iff \pi_0 < \frac{p_1}{p_0+p_1}.$$

On the other hand, we choose $H_0$ if

$$k \leq \frac{\log(\tau p_0/p_1)}{\log\left(\frac{1-p_1}{1-p_0}\right)} + 1, \qquad \forall k.$$

It is clear that this condition cannot be satisfied since no matter what the right-hand side is, there exists a $k$ larger than the value of the right-hand side.

# Problem 7

The probability that a light bulb manufactured by Transylvania Corp. burns out during the $n$th hour of operation is $p_1(n), n = 1, 2, \cdots$. The probability that a light bulb manufactured by Eastinghouse Corp. burns out during the $n$th hour of operation is $p_2(n), n = 1, 2, \cdots$. A bulb is equally likely to have been made by one of the two manufacturers.

a) What is the probability that the bulb burns out during the $M$th hour of operation?

b) Given that the bulb burned out during the $M$th hour of operation, what is the probability that it was manufactured by Transylvania Corp.?

c) Given that the bulb is still burning at the end of the $M$th hour of operation, what is the probability that it was manufactured by Transylvania Corp.?

# Solution

Let BM denote the bulb manufacturer and,
$A$ : bulb burns out during $M$th hour
$T$ : Bulb manufacturer is Transylvania Corp.
$E$ : Bulb manufacturer is Eastinghouse Corp.

a) Since the blubs are equally likely to be manufacutered by Transylvania Corp and Eastinghouse Corp., $P(T) = P(E) = \frac{1}{2}$ . Hence, the probability that the bulb burns out during $M$th hour is

$$\begin{aligned} P(A) &= \frac{1}{2}\left(P(A|T) + P(A|E)\right) \\ &= \frac{1}{2}\left(p_1(M) + p_2(M)\right) \end{aligned}$$

b) Given that the bulb burned out during the $M$th hour of operation, the probability that it was manufactured by Transylvania Corp. is

$$\begin{aligned} P(T|A) &= \frac{P(AT)}{P(A)} \\ &= \frac{P(A|T)P(T)}{P(A)} \\ &= \frac{p_1(M)}{p_1(M) + p_2(M)} \end{aligned}$$

c) Let $C$ denote the event that the Bulb is still buring at the end of $M$th hour. Then,

$$\begin{aligned} P(C|T) &= P(\text{Bulb is still buring at end of M-th hour}|T) \\ &= \sum_{i=M+1}^{\infty} p_1(i) \\ P(C|E) &= P(\text{Bulb is still buring at end of M-th hour}|E) \\ &= \sum_{i=M+1}^{\infty} p_2(i) \end{aligned}$$

Hence, the probability that the bulb was manufactured by Transylvania Corp. given that it is still burning at the end of the M-th hour of operation is

$$\begin{aligned} P(T|C) &= \frac{P(C|T)P(T)}{P(C)} \\ &= \frac{\sum_{i=M+1}^{\infty} p_1(i)\frac{1}{2}}{\sum_{i=M+1}^{\infty} p_1(i)\frac{1}{2} + \sum_{i=M+1}^{\infty} p_2(i)\frac{1}{2}} \\ &= \frac{\sum_{\{i=M+1\}}^{\infty} p_1(i)}{\sum_{i=M+1}^{\infty} p_1(i) + p_2(i)}. \end{aligned}$$

# Problem 8

Alice and Bob play the following game. First, Alice rolls a fair die and then Bob rolls the fair die. If Bob rolls a number at least as large as Alice's number, he wins the game. But if Bob rolled a number smaller than Alice's number, then Alice rolls the die again. If her second roll gives her a number that is less than or equal to Bob's number, the game ends with no winner (a tie, or draw as the British call it). If her second roll gives a number larger than Bob's number, Alice wins the game. Find the probability that Alice wins the game and the probability that Bob wins the game. Also, find the probability of a tie directly (and not as P(tie) = 1-P(Alice wins)-P(Bob wins).) If the three probabilities do not add up to 1, explain.

# Solution

Let $A$ denote the number Alice rolls, $B$ denote the number Bob rolls, and $C$ denote the number Alice rolls in case Bob does not win in the first round. Hence,

Bob wins if $B \geq A$.
Alice wins if $B < A$ and $B < C$.
It's a tie if $B < A$ and $B \geq C$.

$$
\begin{aligned}
P(B \geq A) &= \sum_{a=1}^{6} P(B \geq a) P(A = a) \\
&= \sum_{a=1}^{6} P(B \geq a) \frac{1}{6} \\
&= \sum_{a=1}^{6} \left( \frac{7-a}{6} \right) \frac{1}{6} \\
&= \frac{21}{36} = \frac{7}{12}
\end{aligned}
$$

Probability that Alice wins is

$$
\begin{aligned}
P(B < A \& B < C) &= \sum_{a=1}^{6} \sum_{c=1}^{6} P(B < a \& B < c) P(A = a, C = c) \\
&= \sum_{a=1}^{6} \sum_{c=1}^{6} P(B < \min(a, c)) \frac{1}{6^2} \\
&= \sum_{a=1}^{6} \sum_{c=1}^{6} \frac{\min(a, c) - 1}{6} \frac{1}{6^2} \\
&= \frac{11 \times 0}{6^3} + \frac{9 \times 1}{6^3} + \frac{7 \times 2}{6^3} + \frac{5 \times 3}{6^3} + \frac{3 \times 4}{6^3} + \frac{1 \times 5}{6^3} \\
&= \frac{55}{216}
\end{aligned}
$$

Probability that there will be a tie is

$$
\begin{aligned}
P(B < A \& B \geq C) &= \sum_{a=1}^{6} \sum_{c=1}^{6} P(B < a \& B \geq c) P(A = a, C = c) \\
&= \sum_{a=1}^{6} \sum_{c=1}^{6} P(B \in [c, a) \& c < a) \frac{1}{6^2} \\
&= \sum_{a=1}^{6} \sum_{c=1}^{6} \left( \frac{a-c}{6} \right)^{+} \frac{1}{6^2} \\
&= \frac{5 \times 1}{6^3} + \frac{4 \times 2}{6^3} + \frac{3 \times 3}{6^3} + \frac{2 \times 4}{6^3} + \frac{1 \times 5}{6^3} \\
&= \frac{35}{216}
\end{aligned}
$$

Hence, the three probabilities sum up to 1.

$$
P(\text{Bob wins}) + P(\text{Alice wins}) + P(\text{tie}) = \frac{21}{36} + \frac{55}{216} + \frac{35}{216} = 1
$$

The three probabilties should sum up to 1 because at the end of the match, these are the three possible results.