

## Problem Set 6

Reading: Discrete RVs, Maximum Likelihood, Inequalities

Quiz Date: Friday, June 29

**Note: It is very important that you solve the problems first and check the solutions afterwards.**

## Problem 1

**(Markov's and Chebyshev's inequalities)** The number of items produced in a factory during a week is a random variable with mean 50.

- What can be said about the probability that this week's production is at least 100?
- If the variance of a week's production is known to equal 25, can we obtain a better bound for part (a)?

## Solution

Let  $X$  denote this week's production.

- $P(X \geq 100) \leq \frac{E[X]}{100} = \frac{50}{100} = \frac{1}{2}$ .
- $P(X \geq 100) \leq P(|X - 50| \geq 50) \leq \frac{25}{50^2} = \frac{1}{100}$ . This shows that if the variance is known, using Chebyshev's inequality may give tighter bounds than Markov's inequality.

## Problem 2

**(The maximum of pmfs of common RVs)** For the following RVs, find where the maximum occurs. Assume  $0 < p < 1$ .

- $Z \sim \text{Geo}(p)$ .
- $S \sim \text{NegBin}(r, p)$ .

## Solution

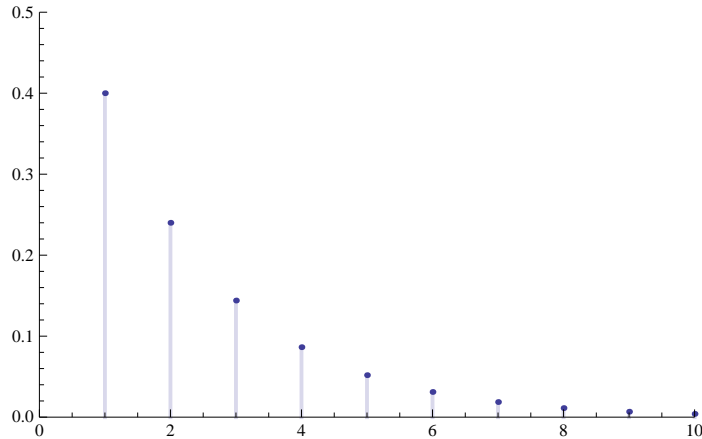
- For  $Z$ , we have

$$p_Z(k) = (1-p)^{k-1} p, \quad k = 1, 2, \dots$$

and

$$\frac{p_Z(k)}{p_Z(k-1)} = (1-p).$$

Assuming  $p > 0$ , we find  $p_Z(k) < p_Z(k-1)$  so the maximum occurs at  $k = 1$ . What is the value of  $p$  in the example below?



b) For  $S$ , the pmf is

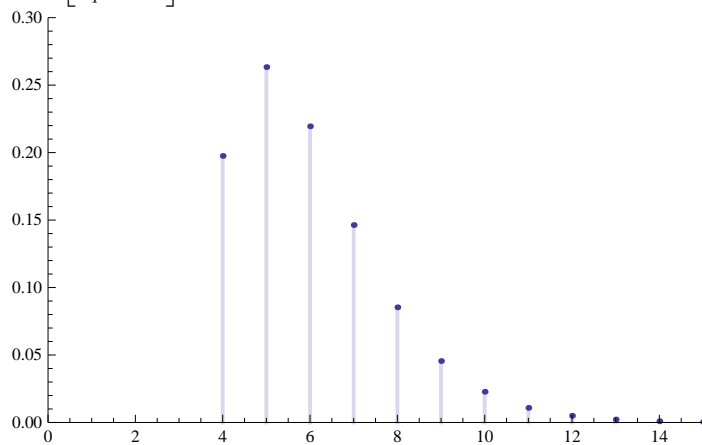
$$p_S(k) = \binom{k-1}{r-1} p^r (1-p)^k, \quad k = r, r+1, \dots$$

so

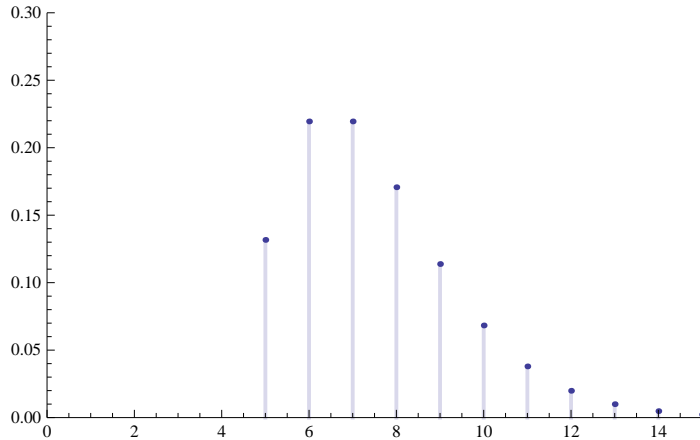
$$\frac{p_S(k)}{p_S(k-1)} = \frac{k-1}{k-r} (1-p),$$

$$p_S(k) > p_S(k-1) \iff k - kp - 1 + p > k - r \iff k < \frac{r-1}{p} + 1.$$

We consider two cases. First, suppose  $\frac{r-1}{p}$  is not an integer. In this case the maximum occurs at  $k = \left\lfloor \frac{r-1}{p} + 1 \right\rfloor$ . In the example below,  $r = 4$  and  $p = 2/3$ .



Finally, if  $\frac{r-1}{p}$  is an integer, the maximum occurs for  $k = \frac{r-1}{p}$  and  $k = \frac{r-1}{p} + 1$ . In the example below,  $r = 3$  and  $p = 2/3$ .



### Problem 3

**(Condence Intervals)** A communication system designer is simulating a communication link. The link is being designed for binary transmission, i.e., bits  $b_k \in \{0, 1\}$  are transmitted, and bits  $\hat{b}_k \in \{0, 1\}$  are received in the  $k$ th bit-period. Noise and other sources of channel impairments result in bit-errors, i.e., the event  $\hat{b}_k \neq b_k$ . The bit errors are assumed to occur independently from one bit-period to the next with a probability of error, or bit error-rate (BER), of  $p_e$ . The designer wishes to estimate  $p_e$  by running  $n$  bits through a simulation model, counting the number of errors  $E$  by comparing the transmitted bits  $b_k$  with recovered bits  $\hat{b}_k$ , and obtaining a BER estimate  $\hat{p}_e = \frac{E}{n}$ , where  $E$  is the error count, i.e., the total number of bits in error in a stream of  $n$  bits.

- Explain why  $E$  can be modeled as a binomial random variable with parameters  $(n, p_e)$
- Suppose that the designer knows that  $p_e \leq 10^{-4}$ . What is the minimum value of  $n$  required to achieve a confidence level of 99% and confidence interval of length  $10^{-5}$ ?

### Solution

- Let  $E_k$  be the event that  $b_k$  is received with an error. We have  $E = \sum_{k=1}^n E_k$ . Since  $E_k$  are independent and identically distributed Bernoulli random variables with parameter  $p_e$ ,  $E$  has a binomial distribution with parameters  $(n, p_e)$ .
- We have

$$P\left(p_e \in \left(\hat{p}_e - a\sqrt{\frac{p_e(1-p_e)}{n}}, \hat{p}_e + a\sqrt{\frac{p_e(1-p_e)}{n}}\right)\right) \geq 1 - \frac{1}{a^2}.$$

Since  $p_e \leq 10^{-4}$ , we find  $p_e(1-p_e) \leq 10^{-4}$  and so

$$P\left(p_e \in \left(\hat{p}_e - \frac{a}{100\sqrt{n}}, \hat{p}_e + \frac{a}{100\sqrt{n}}\right)\right) \geq 1 - \frac{1}{a^2}.$$

Confidence level of 99% requires  $a$  to equal 10. We must then have

$$\frac{2 \times 10}{100\sqrt{n}} = 10^{-5} \Rightarrow n = 4 \times 10^8.$$

## Problem 4

**(Maximum-Likelihood Estimation and Confidence Intervals)** An urn contains 11 red balls and an unknown number of blue balls. Let the total number of balls in the urn be denoted by  $t$ . The experiment consists of drawing one ball at random from the urn and noting its color. Consider 100 independent trials of this experiment; the ball drawn is replaced, and the urn shaken well before the next ball is drawn.

- If it is observed that 20 of the drawings resulted in a red ball and 80 in a blue ball, what is the maximum likelihood estimate  $\hat{t}_{ML}$  of the total number of balls  $t$  in the urn?
- Let the number of red balls in 100 drawings be denoted by  $Y$ . Consider the estimate  $\hat{t} = \frac{1100}{Y}$  of the total number  $t$  of balls in the urn. Find a confidence interval expression for  $\hat{t}$  and  $t$ .

## Solution

- The number of red balls observed has a binomial distribution with parameters  $(100, \frac{11}{t})$ . So the likelihood function is

$$l(t) = \binom{100}{80} \left(\frac{t-11}{t}\right)^{80} \left(\frac{11}{t}\right)^{20}$$

which is maximized when  $t$  satisfies the equation  $\frac{11}{t} = \frac{20}{100}$  and thus

$$\hat{t}_{ML} = 55.$$

(How would we solve the problem if the solution to the equation above was not an integer?)

- We have that  $EY = 100 \cdot \frac{11}{t} = \frac{1100}{t}$ . Using Chebyshev's inequality,

$$P\left(\left|Y - \frac{1100}{t}\right| \geq a\sigma\right) \leq \frac{1}{a^2}$$

where  $\sigma^2 = \text{Var}[Y]$ . Since  $Y$  has a binomial distribution,  $\sigma^2 \leq 100 \left(\frac{1}{2}\right) \left(\frac{1}{2}\right)$  or  $\sigma \leq 5$ . So

$$P\left(\left|Y - \frac{1100}{t}\right| \geq 5a\right) \leq \frac{1}{a^2}.$$

Since  $\hat{t} = \frac{1100}{Y}$ , we obtain

$$\begin{aligned} P\left(\left|\frac{1}{\hat{t}} - \frac{1}{t}\right| \geq \frac{a}{220}\right) &\leq \frac{1}{a^2}, \\ P\left(\frac{1}{t} \in \left(\frac{1}{\hat{t}} - \frac{a}{220}, \frac{1}{\hat{t}} + \frac{a}{220}\right)\right) &\geq 1 - \frac{1}{a^2}, \\ P\left(t \in \left(\frac{\hat{t}}{1 - a\hat{t}/220}, \frac{\hat{t}}{1 + a\hat{t}/220}\right)\right) &\geq 1 - \frac{1}{a^2}. \end{aligned}$$

This confidence interval expression is not entirely useful as the length of the interval is *random*.

## Problem 5

Suppose  $X$  is a negative binomial random variable with parameters  $r$  and  $p$ , where  $r$  is unknown but  $p$  is given. Find the ML estimate of  $r$  if  $X = k$  is observed.

## Solution

The likelihood function is

$$l(r) = \binom{k-1}{r-1} p^r (1-p)^{k-r}.$$

We find the ration  $\frac{l(r)}{l(r-1)}$ .

$$\begin{aligned} \frac{l(r)}{l(r-1)} &= \frac{\binom{k-1}{(k-r)!(r-1)!}}{\binom{k-1}{(k-r+1)!(r-2)!}} \frac{p}{1-p} \\ &= \frac{k-r+1}{r-1} \frac{p}{1-p}, \end{aligned}$$

Letting  $\frac{l(r)}{l(r-1)} > 1$  yields

$$\frac{k-r+1}{r-1} \frac{p}{1-p} > 1 \iff kp - rp + p > r - rp - 1 + p \iff kp > r - 1 \iff r < kp + 1$$

So if  $r < kp + 1$ , then  $l(r) > l(r-1)$ . If  $r = kp + 1$ , then  $l(r) = l(r-1)$ , and if  $r > (kp + 1)$ , then  $l(r) < l(r-1)$ . From this, if  $kp$  is not an integer, then  $\hat{r}_{ML} = \lfloor kp + 1 \rfloor$  and if  $kp$  is an integer  $\hat{r}_{ML} \in \{kp, kp + 1\}$ .

## Problem 6

In this problem you are allowed to use a computer to find numerical values.

A newsboy purchases 50 newspapers for 35 cents each and sells them for 60 cents each. He can recycle any unsold papers and recoup 25 cents for each. A total of 100 people pass by him each day, and each decides (independently of all other passers by) to ask to buy a paper with probability 0.6. Let  $X$  denote the number of papers sold.

- What is the probability that the newsboy sells all 50 papers?
- Let  $Z$  denote the daily profit (in cents) that the newsboy makes. What is his maximum profit? What is his maximum loss? More generally, express  $Z$  in terms of  $X$ .
- Compute the newsboy's average daily profit.
- One day, after the newsboy has sold all of his papers, an angry customer demands a paper. After this experience, the newsboy decides to increase the number of papers he buys. Being fairly cautious, he decides to begin by buying one extra paper. What is the probability that he sells this extra paper? What is the average additional profit that he makes from this paper?
- Encouraged by his success, on the next day he decides to buy 52 papers. What is the probability that he sells this extra paper (i.e., the 52nd paper)? What is the average additional profit that he makes from this 52nd paper? (Note that you are now being asked to compute his average additional profit from the 52nd paper, over and above the profit from 51 papers that he bought the previous day). Is the average additional profit from the 52nd paper larger or smaller than the average additional profit from the 51st paper?

## Solution

- Let  $D$  be the RV that denotes the number of people who decide to buy a paper. Clearly  $D \sim \text{Bin}(100, .6)$ . All papers are sold when  $D \geq 50$ .

$$P(X = 50) = P(D \geq 50) = \sum_{i=50}^{100} \binom{100}{i} .6^i .4^{100-i} = 0.983238$$

where the sum can be computed with MATLAB/Mathematica.

b) We have

$$\begin{aligned} Z &= 50 \times (-35) + X \times 60 + (50 - X) \times 25 \\ &= 35X - 500 \end{aligned}$$

Maximum profit is for  $X = 50 \rightarrow Z = 1250$ .

Maximum loss is for  $X = 0 \rightarrow Z = -500$ .

c) The average profit is  $EZ = 35EX - 500$  so we need to find  $EX$ . Note that, for  $0 \leq i \leq 49$ ,

$$P(X = i) = P(D = i) = \binom{100}{i} .6^i .4^{100-i}.$$

Hence,

$$\begin{aligned} EX &= \sum_{i=0}^{50} ip_X(i) \\ &= \sum_{i=0}^{49} ip_X(i) + 50p_X(50) \\ &= \sum_{i=0}^{49} i \binom{100}{i} .6^i .4^{100-i} + 50 \times 0.983238 \\ &= 49.9609 \end{aligned}$$

and so  $EZ = 1248.63$ .

d) The probability that he sells the 51st paper is

$$\sum_{i=51}^{100} \binom{100}{i} .6^i .4^{100-i} = 0.972901$$

and the average additional profit is

$$0.972901 \times 25 + (1 - 0.972901) \times (-10) = 24.0515$$

since he earns 25 cents if he sells the paper and loses 10 cents if he does not sell the paper.

e) The probability that he sells the 52nd paper is

$$\sum_{i=52}^{100} \binom{100}{i} .6^i .4^{100-i} = 0.957699$$

and the average additional profit is

$$0.957699 \times 25 + (1 - 0.957699) \times (-10) = 23.5195.$$

One can numerically show that the optimum number of papers that the news boy should by is 63.

## Problem 7

We say that event  $B$  gives *positive information* about event  $A$  if  $P(A|B) > P(A)$ , that is, the occurrence of  $B$  makes the occurrence of  $A$  more likely. Now suppose that  $B$  gives positive information about  $A$ . If so

- Does  $A$  give positive information about  $B$ ?
- Does  $B^c$  give negative information about  $A$ , that is, is it true that  $P(A|B^c) < P(A)$ ?
- Does  $B^c$  give positive information or negative information about  $A^c$ ?

## Solution

a) Note that  $P(A)P(B|A) = P(B)P(A|B)$ . Assuming  $P(A), P(B) > 0$ ,

$$\frac{P(A|B)}{P(A)} = \frac{P(B|A)}{P(B)}.$$

Thus  $\frac{P(A|B)}{P(A)} > 1$ , indicates that  $\frac{P(B|A)}{P(B)} > 1$ , and so, the answer is yes.

b) Let

$$\begin{aligned} P(A \cap B) &= x, \\ P(A \cap B^C) &= y, \\ P(A^C \cap B) &= z. \end{aligned}$$

We have that  $P(A|B) > P(A)$ , and thus

$$\frac{x}{x+z} > x+y \iff (x+y)(x+z) < x. \quad (1)$$

We need to show (or disprove) that

$$\begin{aligned} P(A|B^C) &= \frac{y}{1-(x+z)} < x+y \\ \iff y &< (x+y) - (x+y)(x+z) \\ \iff (x+y)(x+z) &< x. \end{aligned}$$

But the last inequality is equivalent to (1) and hence  $B^C$  gives negative information about  $A$ .

c) We have  $P(A^C|B^C) = 1 - P(A|B^C)$  and since  $P(A|B^C) < P(A)$ , we can write

$$P(A^C|B^C) > 1 - P(A) = P(A^C).$$

Thus  $B^C$  gives positive information about  $A^C$ .