ECE313		**Summer 2012**
	Problem Set 5	
Reading: Independence, Important Discrete RVs		Quiz Date: No Quiz, Exam on Tuesday, 26

Note: It is very important that you solve the problems first and check the solutions afterwards.

Problem 1

(Independent and dependent events in a Karnaugh map) Mad scientist Sheila Frankenstein has created a monster. Let A be the event "Monster eats roses," and let B be the event "Monster reads Shakespeare."

a) Suppose $P(A) = 1/3$, $P(B) = 1/4$, and the events A and B are independent. Draw a Karnaugh map of the experimental outcomes. Fill in the probabilities $P(AB)$, $P(AB^c)$, $P(A^cB)$, and $P(A^cB^c)$.

b) Suppose $P(A) = 1/3$, $P(B) = 1/4$, and $P(B|A) = 1/8$. Draw a Karnaugh map of the experimental outcomes. Fill in the probabilities $P(AB)$, $P(AB^c)$, $P(A^cB)$, and $P(A^cB^c)$.

Solution

a) Solution should be a Karnaugh map showing $P(AB) = 1/12$, $P(AB^c) = 1/4$, $P(A^cB) = 1/6$, and $P(A^cB^c) = 1/2$.

b) Solution should be a Karnaugh map showing $P(AB) = 1/24$, $P(AB^c) = 7/24$, $P(A^cB) = 5/24$, and $P(A^cB^c) = 11/24$.

Problem 2

Let X be a geometric random variable with parameter $1/4$ and let $Y = \sin(\pi X/2)$. Find the pmf of Y.

Solution

Since the set of possible values for X is the set of positive integers, the set of possible values for Y is $\{0, 1, -1\}$. Namely,

$$Y = \begin{cases} 0, & \text{if } X = 2k, \\ 1, & \text{if } X = 4k+1, \\ -1, & \text{if } X = 4k+3. \end{cases}$$

So

$$p_Y(0) = \sum_{k=1}^{\infty} p_X(2k) = \sum_{k=1}^{\infty} \frac{1}{4}\left(\frac{3}{4}\right)^{2k-1} = \frac{3}{16}\sum_{k=0}^{\infty}\left(\frac{9}{16}\right)^k = \frac{3}{7},$$

$$p_Y(1) = \sum_{k=1}^{\infty} p_X(4k+1) = \sum_{k=0}^{\infty} \frac{1}{4}\left(\frac{3}{4}\right)^{4k} = \frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{81}{256}\right)^k = \frac{64}{175},$$

$$p_Y(-1) = \sum_{k=1}^{\infty} p_X(4k+3) = \sum_{k=0}^{\infty} \frac{1}{4}\left(\frac{3}{4}\right)^{4k+2} = \frac{9}{64}\sum_{k=0}^{\infty}\left(\frac{81}{256}\right)^k = \frac{36}{175}.$$

Problem 3

Let X be a random variable representing the number of times one must throw a die until the outcomes 1 or 2 have occurred four times. Find the pmf, the expected value, and the variance of X.

Solution

Let us call the die landing on 1 or 2 a success. The experiment consists of a sequence of independent Bernoulli experiments with success probability $1/3$ and the random variable X denotes the number of times until success has occurred four times. Hence, X has a negative binomial distributions with parameters $r = 4$ and $p = 1/3$. We have

$$p_X(n) = \binom{n-1}{3}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^{n-4}$$

for $n = 4, 5, 6, \cdots$. Additionally, $E[X] = \frac{r}{p} = 12$, and $\text{Var}[X] = \frac{r(1-p)}{p^2} = 24$.

Problem 4

Two fair dice are rolled. Let D be the event that the dice land on different numbers and S be the event that at least one lands on 6. What is $P(S|D)$? Are S and D independent?

Solution

$$P(S|D) = \frac{P(S \cap D)}{P(D)} = \frac{|S \cap D|}{|D|} = \frac{1 \times 5 + 5 \times 1}{6 \times 5} = \frac{1}{3}.$$

$$P(S) = \frac{11}{36}.$$

Since $\frac{11}{36} \neq \frac{1}{3}$, the events are not independent.

Problem 5

Example 2.4.9 of the text.

Problem 6

Suppose $X_1, X_2, \cdots, X_n$ are independent discrete random variables. Assume that each X_i has mean μ_i and variance σ_i^2. Let

$$X = \sum_{i=1}^{n} X_i$$

and show that

$$\boxed{\text{Var}[X] = \sum_{i=1}^{n} \text{Var}[X_i] = \sum_{i=1}^{n} \sigma_i^2.}$$

If, in addition, all X_i have the same distribution, $X_1, X_2, \cdots, X_n$ is called a sequence of *independent, identically distributed (iid)* random variables. Suppose each X_i has mean μ and variance σ^2. What is $\text{Var}[X]$ in this case?

Solution

First we find $E[X]$.

$$\begin{aligned} E[X] &= E\left[\sum_{i=1}^{n} X_i\right] \\ &= \sum_{i=1}^{n} E[X_i] \\ &= \sum_{i=1}^{n} \mu_i. \end{aligned}$$

Now the variance can be written as

$$\begin{aligned} \mathrm{Var}[X] &= E\left[(X - E[X])^2\right] \\ &= E\left[\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu_i\right)^2\right] \\ &= E\left[\left(\sum_{i=1}^{n} (X_i - \mu_i)\right)^2\right] \\ &= E\left[\left(\sum_{i=1}^{n} (X_i - \mu_i)\right)\left(\sum_{j=1}^{n} (X_j - \mu_j)\right)\right] \\ &= E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} (X_i - \mu_i)(X_j - \mu_j)\right] \\ &= E\left[\sum_{i=1}^{n} (X_i - \mu_i)^2 + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} (X_i - \mu_i)(X_j - \mu_j)\right] \\ &= \sum_{i=1}^{n} E\left[(X_i - \mu_i)^2\right] + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} E[(X_i - \mu_i)(X_j - \mu_j)] \\ &= \sum_{i=1}^{n} \sigma_i^2 \end{aligned}$$

where we have used the fact that $E[(X_i - \mu_i)(X_j - \mu_j)] = 0$ for $i \neq j$. This follows from the fact that $X_i - \mu_i$ and $X_j - \mu_j$ are independent since X_i and X_j are independent. (In general, if random variables Y and Z are independent, $f(Y)$ and $g(Z)$ are also independent for any functions f, g.) So

$$\begin{aligned} E[(X_i - \mu_i)(X_j - \mu_j)] &= E[X_i - \mu_i]\, E[X_j - \mu_j] \\ &= 0 \cdot 0 \\ &= 0. \end{aligned}$$

If X_i are *iid*, then $\mathrm{Var}[X] = n\sigma^2$.

Problem 7

(The maximum of pmfs of common RVs) For the following RVs, find where the maximum occurs. Assume $0 < p < 1$ and $\lambda > 0$.

a) $X \sim \text{Bin}(n, p)$.

b) $Y \sim \text{Poi}(\lambda)$.

Solution

a) The pmf of X is

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, 2, \cdots, n.$$

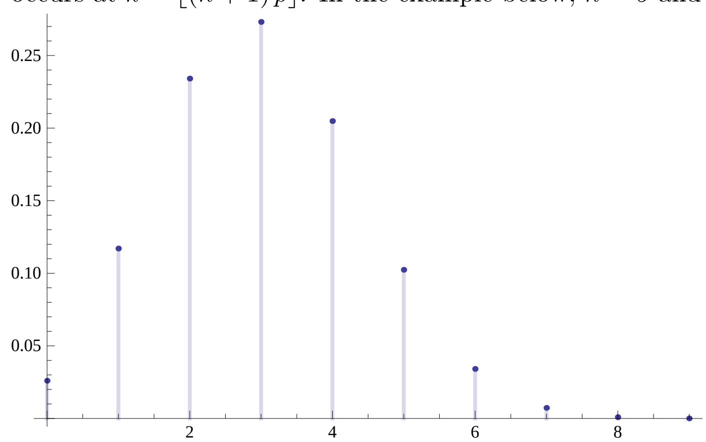
We look for the value of k that maximizes $p_X(k)$. Since the argument k is a discrete argument, we cannot use differentiation. Instead, we find the ratio $\frac{p_X(k)}{p_X(k-1)}$.

$$\begin{aligned} \frac{p_X(k)}{p_X(k-1)} &= \frac{\left(\frac{n!}{k!(n-k)!}\right) p^k (1-p)^{n-k}}{\left(\frac{n!}{(k-1)!(n-k+1)!}\right) p^{k-1} (1-p)^{n-k+1}} \\ &= \frac{n-k+1}{k} \frac{p}{1-p} \end{aligned}$$

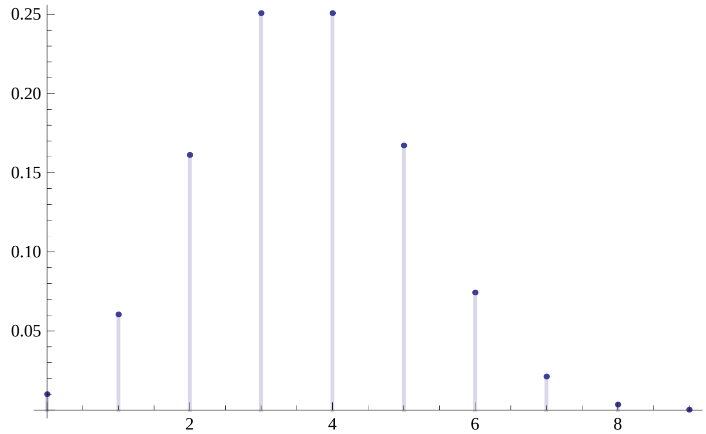
Next, we find values of k such that $\frac{p_X(k)}{p_X(k-1)} > 1$.

$$\frac{n-k+1}{k} \frac{p}{1-p} > 1 \iff np - kp + p > k - kp \iff k < (n+1)p$$

So $p_X(k) > p_X(k-1)$ if $k < (n+1)p$. Similarly, $p_X(k) = p_X(k-1)$ if $k = (n+1)p$ and $p_X(k) < p_X(k-1)$ if $k > (n+1)p$.
First, let us assume $(n+1)p$ is not an integer. In this case, based on the discussion above, the maximum occurs at $k = \lfloor (n+1)p \rfloor$. In the example below, $n = 9$ and $p = 1/3$.



Second, let us assume $(n+1)p$ is an integer. Then, two values of k, $(n+1)p - 1$ and $(n+1)p$, attain the maximum. In the example below, $n = 9$ and $p = 2/5$.



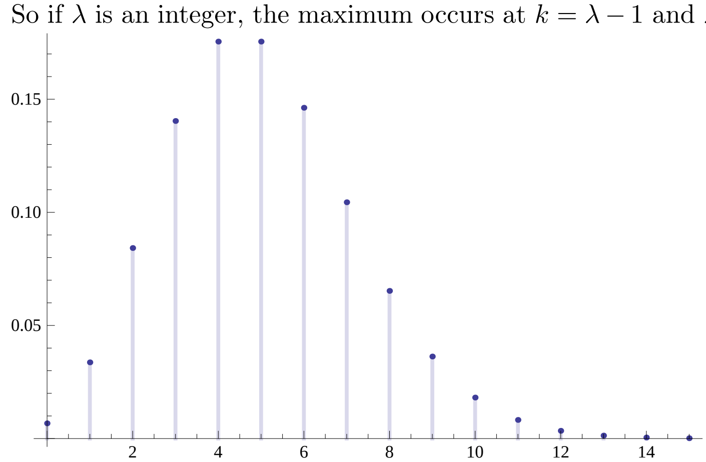
b) The pmf of Y is

$$p_Y(k) = e^{-\lambda}\lambda^k/k!, \qquad k = 0, 1, 2, \cdots$$

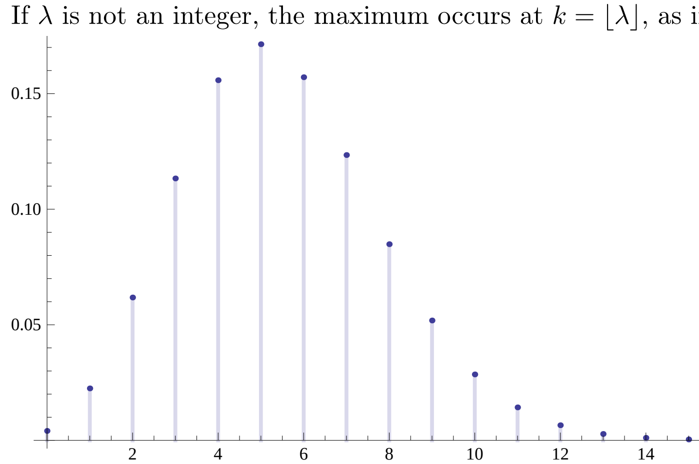
Similar to the previous part:

$$\frac{p_Y(k)}{p_Y(k-1)} = \frac{\lambda}{k}$$

So if λ is an integer, the maximum occurs at $k = \lambda - 1$ and λ, as in the following example where $\lambda = 5$.



If λ is not an integer, the maximum occurs at $k = \lfloor \lambda \rfloor$, as in the example below where $\lambda = 5.5$.



Problem 8

Suppose that eight bits are used to encode a standard ASCII character: seven bits determine the number corresponding to the ASCII character and the eighth bit is a checksum bit set to the XOR of the seven other bits. Suppose that these bits are transmitted on an optical fiber, and that the probability of an error in any bit transmission is $p = 0.001$. The receiver detects an error whenever the XOR of the bits of a received 8-bit word is non-zero.

a) Specify the pmf for the random variable X that denotes the number of error bits in an eight-bit word.

b) Compute the probability that a received eight-bit word will contain undetected errors. You may find it helpful to consider the following questions. If exactly one error occurs, what will be the XOR of the received bits? If exactly two errors occur, what will be the XOR of the received bits? If exactly three errors occur, what will be the XOR of the received bits? ... In which cases will the occurrence of errors be undetected using the simple XOR test?

c) Suppose N eight-bit words are transmitted. Determine the pmf for the random variable Y that denotes the number received words that contain undetected errors.

Solution

a) Let X_i be a Bernoulli random variable where

$$X_i = \begin{cases} 1, & \text{if } i\text{th bit is erroneous} \\ 0, & \text{eles.} \end{cases}$$

Then, $X = \sum_{i=1}^{8} X_i$. Since X_i are independent and have the same parameter $p = 10^{-3}$, X is a binomial RV with parameters $n = 8$ and $p = 10^{-3}$. Thus, for $1 \leq k \leq 8$,

$$p_X(k) = \binom{8}{k} .001^k .999^{8-k}$$

b) If the number of errors is odd, the receiver will observe that the XOR of the bits is 1 and detects an error. On the other hand, if an even number of errors occur, the error is undetected. Thus, the probability of undetected errors in an 8-bit word, p_w, is

$$p_w = \sum_{i=2,4,6,8} p_X(i) = 0.0000278325$$

Note that this probability is very close to $p_X(2) = 0.0000278324$. What does this tell you?

c) A word contains undetected error with probability $p_w = 0.0000278325$, as obtained in part (b). The number of words containing undetected errors is a random variable with a binomial distribution with parameters N and p_w. For $1 \leq k \leq N$,

$$p_Y(k) = \binom{N}{k} p_w^k (1 - p_w)^{N-k}.$$