

Problem Set 4

Reading: RVs, mean, variance, and conditional probability

Quiz Date: Friday, June 22

Note: It is very important that you solve the problems first and check the solutions afterwards.

Problem 1

Consider a random variable X with the following pmf:

$$p_X(i) = c(1-p)^{i-1}, \quad \text{for } i = 1, 2, \dots$$

Find the value of c such that $p_X(i)$ is a valid pmf. Furthermore,

- Find EX and $\text{Var}[X]$.
- If it is known that $E[2X + 3] = 9$, what is p ?
- If it is known that $\text{Var}[2X + 3] = 24$, what is p ?

Solution

For the pmf to be valid we need $\sum_{i=1}^{\infty} p_X(i) = 1$. So

$$\begin{aligned} \sum_{i=1}^{\infty} p_X(i) &= c \sum_{i=1}^{\infty} (1-p)^{i-1} = c \sum_{j=0}^{\infty} (1-p)^j \\ &= c \cdot \frac{1}{1-(1-p)} = \frac{c}{p} = 1 \end{aligned}$$

where we have used the fact that

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \quad \text{for } |x| < 1.$$

This implies that we must have $c = p$.

Before obtaining the mean and the variance, let us obtain two useful formulas. Take the derivative with respect to x on both sides of $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$ to obtain

$$\sum_{i=1}^{\infty} ix^{i-1} = \frac{1}{(1-x)^2}.$$

Differentiation for a second time yields

$$\sum_{i=2}^{\infty} i(i-1)x^{i-2} = \frac{2}{(1-x)^3}.$$

a) For the mean we have

$$\begin{aligned} E[X] &= \sum_{i=1}^{\infty} i p_X(i) = \sum_{i=1}^{\infty} i (1-p)^{i-1} p \\ &= p \sum_{i=1}^{\infty} i (1-p)^{i-1} = p \cdot \frac{1}{p^2} = \frac{1}{p} \end{aligned}$$

To obtain the variance, first we find $E[X^2]$:

$$\begin{aligned} E[X^2] &= \sum_{i=1}^{\infty} i^2 p_X(i) \\ &= \sum_{i=1}^{\infty} (i^2 - i + i) (1-p)^{i-1} p \\ &= \sum_{i=2}^{\infty} (i^2 - i) (1-p)^{i-1} p + \sum_{i=1}^{\infty} i (1-p)^{i-1} p \\ &= p(1-p) \sum_{i=2}^{\infty} i(i-1) (1-p)^{i-2} + \frac{1}{p} \\ &= p(1-p) \frac{2}{p^3} + \frac{1}{p} \\ &= \frac{2-p}{p^2} \end{aligned}$$

The variance is $\text{Var}[X] = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$.

b) $E[2X + 3] = 2E[X] + 3 = 9 \Rightarrow E[X] = 3 \Rightarrow p = 1/3$.

c) $\text{Var}[2X + 3] = 4\text{Var}[X] = 24 \Rightarrow \text{Var}[X] = 6 \Rightarrow \frac{1-p}{p^2} = 6$. The last equality is satisfied if $6p^2 + p - 1 = 0$. The solutions to this equation are $\{-1/2, 1/3\}$. Only $1/3$ is acceptable (why?).

Problem 2

Example 2.2.5 of the text.

Problem 3

Suppose that you and nine of your friends are planning on going to a concert, four of your friends are also taking ECE 313, and five are not. Unfortunately, your car only fits five people so you randomly choose four friends to go with you (all friends are equally likely to be chosen), and the rest will miss the concert. When you get there, it turns out that those who are taking ECE 313 get in for free, and everyone else pays \$10. Let X be the total amount of money you and your friends end up paying to get into the concert.

- What values can X take?
- Find the pmf of X .
- Find the expected value of X .
- If instead of getting in for free, the ECE students have to pay z dollars, what is the expected value of the money that you have to pay?

Solution

- a) Let Y be the number of non-ECE313 students that go with you, then $Y \in \{0, 1, 2, 3, 4\}$ and $X = 10Y$. Therefore, $X \in \{0, 10, 20, 30, 40\}$.
- b) You choose four of your nine friends to ride with you, so that $|\Omega| = \binom{9}{4} = 126$. Let us find the size of the event $\{Y = i\}$ for $i \in \{0, 1, 2, 3, 4\}$: There are $\binom{5}{i}$ ways to choose i non-ECE 313 students from a set of 5 non-ECE 313 students, and there are $\binom{4}{4-i}$ ways to choose the remaining students from the ECE 313 friends. So $|\{Y = i\}| = \binom{5}{i} \binom{4}{4-i}$ and thus

$$p_X(k) = P\{Y = k/10\} = \frac{\binom{5}{k/10} \binom{4}{4-k/10}}{126}, \quad \text{for } k \in \{0, 10, 20, 30, 40\}$$

or equivalently,

$$\begin{aligned} p_X(0) &= P\{Y = 0\} = \frac{\binom{5}{0} \binom{4}{4}}{126} = \frac{1}{126}, \\ p_X(10) &= P\{Y = 1\} = \frac{\binom{5}{1} \binom{4}{3}}{126} = \frac{20}{126}, \\ p_X(20) &= P\{Y = 2\} = \frac{\binom{5}{2} \binom{4}{2}}{126} = \frac{60}{126}, \\ p_X(30) &= P\{Y = 3\} = \frac{\binom{5}{3} \binom{4}{1}}{126} = \frac{40}{126}, \\ p_X(40) &= P\{Y = 4\} = \frac{\binom{5}{4} \binom{4}{0}}{126} = \frac{5}{126}. \end{aligned}$$

c) $E[X] = \sum_{u_i} u_i p_X(u_i) = 0 \left(\frac{1}{126}\right) + 10 \left(\frac{20}{126}\right) + 20 \left(\frac{60}{126}\right) + 30 \left(\frac{40}{126}\right) + 40 \left(\frac{5}{126}\right) = \frac{200}{9}$

- d) Let \hat{X} be a random variable denoting the new amount to be paid. Then,

$$\hat{X} = 10Y + z(5 - Y) = X \left(1 - \frac{z}{10}\right) + 5z.$$

Therefore, $E[\hat{X}] = E[X] \left(1 - \frac{z}{10}\right) + 5z = \frac{200}{9} \left(1 - \frac{z}{10}\right) + 5z$.

Problem 4

Let C be a real-valued constant, and let $n \geq 1$ be an integer. The pmf for a discrete-type random variable X is given by $p_X(i) = Ci$ for integers $1 \leq i \leq n$ and zero else. Some helpful summation identities can be found in Appendix 6.2 of the course notes.

- a) Find the constant C such that p_X is a valid pmf.
- b) Find $E[X]$.
- c) Find $E\left[1 + \frac{1}{X}\right]$.

Solution

a) The pmf has to sum up to one, therefore

$$1 = \sum_{u_i} p_X(u_i) = \sum_{i=1}^n Ci = C \sum_{i=1}^n i = C \frac{n(n+1)}{2},$$

and so $C = \frac{2}{n(n+1)}$.

b) By definition,

$$E[X] = \sum_{u_i} u_i p_X(u_i) = \frac{2}{n(n+1)} \sum_{i=1}^n i^2 = \frac{2}{n(n+1)} \frac{n(n+1)(2n+1)}{6} = \frac{2n+1}{3}.$$

c) We have $E\left[1 + \frac{1}{X}\right] = 1 + E\left[\frac{1}{X}\right]$. Using LOTUS,

$$\begin{aligned} E\left[\frac{1}{X}\right] &= \sum_{u_i} \left(\frac{1}{u_i}\right) p_X(u_i) \\ &= \frac{2}{n(n+1)} \sum_{i=1}^n i \left(\frac{1}{i}\right) \\ &= \frac{2}{n(n+1)} \sum_{i=1}^n 1 \\ &= \frac{2}{n(n+1)} \cdot n \\ &= \frac{2}{n+1}. \end{aligned}$$

Hence $E\left[1 + \frac{1}{X}\right] = \frac{n+3}{n+1}$.

Problem 5

To solve this problem, use the general form of the rule $P(AB) = P(A)P(B|A)$ that was discussed in class. The general form is

$$P(A_1 A_2 \cdots A_n) = P(A_1) P(A_2|A_1) P(A_3|A_1 A_2) \cdots P(A_n|A_1 A_2 \cdots A_{n-1}).$$

An ordinary deck of 52 playing cards is randomly divided into 4 piles of 13 cards each. What is the probability that each pile has exactly 1 ace?

Solution

Define events in E_i , $i = 1, 2, 3, 4$ as follows:

$E_1 = \{\text{the ace of spades is in any of the piles}\},$

$E_2 = \{\text{the ace of spades and the ace of hearts are in different piles}\},$

$E_3 = \{\text{the ace of spades, the ace of hearts, and the ace of diamonds are in different piles}\},$

$E_4 = \{\text{all four aces are in different piles}\}.$

The desired probability is $P(E_4) = P(E_1E_2E_3E_4)$ since $E_4 = E_1E_2E_3E_4$. By the given rule

$$P(E_1E_2E_3E_4) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)P(E_4|E_1E_2E_3).$$

Now, $P(E_1) = 1$. Since the pile containing the ace of spades receives 12 of the remaining 51 cards, the ace of hearts must be one of the other 39 cards. We have

$$P(E_2|E_1) = \frac{39}{51}.$$

Similarly, the piles containing the aces of spades and hearts will receive 24 of the remaining 50 cards. So the ace of diamonds must be one of the other 26 cards:

$$P(E_3|E_1E_2) = \frac{26}{50}.$$

Finally,

$$P(E_4|E_1E_2E_3) = \frac{13}{49}.$$

Hence, the probability that each hands has exactly one ace is

$$\frac{39 \cdot 26 \cdot 13}{51 \cdot 50 \cdot 49} \approx 0.105.$$

Problem 6

We randomly choose 5 cards out of a deck of 52 cards.

- What is the probability of full house given that the first card and the second card have different numbers?
- What is the probability of full house given that the first card and the second card have the same number?
- Should the sum of the answers in the previous parts be one?

Solution

- It does not matter what the actual numbers are so for definiteness we assume the first card is 1C and the second card is 2C. That is,

$$P(\text{full house} | \text{1st and 2nd cards have different numbers}) = P(\text{full house} | \text{1st}=1C, \text{2nd}=2C).$$

Now, we have two cases. Number 1 appears twice and 2 appears three times or vice versa. So

$$P(\text{full house} | \text{1st}=1C, \text{2nd}=2C) = \frac{2 \binom{3}{1} \binom{3}{2}}{\binom{50}{3}} = \frac{18 \cdot 3 \cdot 2}{50 \cdot 49 \cdot 48} = \frac{9}{9800} \approx 0.000918367.$$

- Similar to the previous part,

$$P(\text{full house} | \text{1st and 2nd cards have same number}) = P(\text{full house} | \text{1st}=1C, \text{2nd}=1D).$$

Again we have two cases. Number 1 appears twice or number 1 appears three times. If 1 appears twice, there are 12 ways to choose the number that appears three times and $\binom{4}{3}$ ways to choose which suits appear. If 1 appears three times, there are $\binom{2}{1}$ ways to choose the suit of the third 1, 12 ways to choose the number that appear twice, and $\binom{4}{2}$ ways to choose the suits of the number that appears twice. So

$$P(\text{full house} | \text{1st}=1C, \text{2nd}=1D) = \frac{12 \binom{4}{3} + \binom{2}{1} 12 \binom{4}{2}}{\binom{50}{3}} = \frac{12 \cdot 16 \cdot 6}{50 \cdot 49 \cdot 48} \approx 0.00979592.$$

- No. We have $P(A|B) + P(A^c|B) = 1$. The probabilities in this problem are $P(A|B)$ and $P(A|B^c)$ so the sum of the probabilities is not necessarily 1.