ECE313 **Summer 2012**

Problem Set 2

Reading: Counting and equally likely outcomes Quiz Date: Friday, June 15

Problem 1

(counting in two ways)

a) How many subsets of $\{1, 2, \cdots, n\}$ are there?

b) How many subsets of $\{1, 2, \cdots, n\}$ with k elements are there?

c) Show that $2^n = \sum_{k=0}^{n} \binom{n}{k}$.

d) Suppose n is even. How many subsets of $\{1, 2, \cdots, n\}$ contain at least one odd number?

Solution

a) Each subset can be viewed to be equivalent to a binary vector $v = (v_1, v_2, \cdots, v_n)$ of length n: if v_i is equal to 1, then i is inculded in the subset and if v_i is 0, then i is not included in the set. For example, for $n = 3$, the vector $(1, 0, 1)$ corresponds to the subset $\{1, 3\}$.
So for counting the number of subsets, we need to count the number of binary vectors of length n. The answer is thus 2^n.

b) There are $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ subsets of size k. Equivalently, there are $\binom{n}{k}$ binary vectors with k ones and $n - k$ zeros.

c) The number of subsets is equal to the sum of the number of subsets of size k for $k = 0, 1, \cdots, n$. Thus, $2^n = \sum_{k=0}^{n} \binom{n}{k}$.

d) There are 2^n subsets. There are $2^{n/2}$ subsets of $\{1, 2, \cdots, n\}$ with no odd elements; these are the subsets of the set $\{2, 4, \cdots, n\}$. Thus, there are $2^n - 2^{n/2}$ subsets of $\{1, 2, \cdots, n\}$ containing at least one element.

Problem 2

Show that

a) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

b) $\sum_{k=0}^{n} r^k \binom{n}{k} = (r + 1)^n$.

c) $\binom{n}{k} k = n \binom{n-1}{k-1}$.

Solution

a) Consider subsets of $[n]$ of size k and group them based on whether they contain 1. There are $\binom{n}{k}$ subsets of size k. Among these, $\binom{n-1}{k}$ do not contain 1 and $\binom{n-1}{k-1}$ contain 1 (why?). So

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

b) This is an immediate result of the equation $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$. To see this, let $a = r$ and $b = 1$.

c) Suppose we want to choose a committee with k members from a group of n individuals and designate one of the members of the committee as the head of the committee. We do this in two ways. The equality follows from the fact that from both ways we should get the same result.

i) Let's first choose the members of the committee and then choose the head. We can choose the members in $\binom{n}{k}$ ways and then we can choose the head in k ways. So in total, we have $\binom{n}{k}k$ options.

ii) For the second method, we first choose the head among the n individuals and then choose $k-1$ individuals to be the members. We can choose the head in n ways and can choose the remaining members in $\binom{n-1}{k-1}$ ways. So in total, we have $n\binom{n-1}{k-1}$ options.

Problem 3

An experiment consists of observing the contents of an 8-bit register. We assume that all 256 byte values are equally likely to be observed.

a) Let A denote the event that the Least Significant Bit (LSB) is a ONE. What is $P(A)$?

b) Let B denote the event that the register contains 5 ONEs and 3 ZEROes. What is $P(B)$?

c) What is $P(A \cup B)$? What is the probability that exactly one of A and B occur, i.e. what is $P(A \oplus B)$?

Solution

a) Any arbitrary 7-digit binary number concatenated with a ONE is a member of event A. Thus, there are 2^7 such numbers out of 2^8 possible 8-bit numbers. Since all numbers are equaly likely,

$$P(A) = \frac{2^7}{2^8} = \frac{1}{2}.$$

b) The number of binary vectors of length 8 with 5 ONEs is $\binom{8}{5}$. So

$$P(B) = \frac{\binom{8}{5}}{2^8} = \frac{56}{258}.$$

c) We have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ so we need to find $P(A \cap B)$. To find $P(A \cap B)$, we first find $|A \cap B|$, that is, the number of binary vectors with LSB equal to ONE and with 5 ONES. Since LSB is fixed, we must choose 4 other bits to be ONES out of the 7 remaining bits. This can be done in $\binom{7}{4}$ ways. So

$$P(A \cap B) = \frac{\binom{7}{4}}{2^8} = \frac{35}{256}$$

and thus

$$\begin{aligned} P(A \cup B) &= \frac{128}{256} + \frac{56}{256} - \frac{35}{256} \\ &= \frac{149}{256}. \end{aligned}$$

The set $A \oplus B$ is defined as $(AB^c) \cup (BA^c)$. Since $(AB^c) \cap (BA^c) = \varnothing$,

$$P(A \oplus B) = P(AB^c) + P(BA^c).$$

Furthermore, we have $(AB) \cup (AB^c) = A$ and $(BA) \cup (BA^c) = B$. So

$$\begin{aligned} P(A \oplus B) &= P(AB^c) + P(BA^c) \\ &= (P(A) - P(AB)) + (P(B) - P(AB)) \\ &= \frac{128}{256} + \frac{56}{256} - 2 \times \frac{35}{256} \\ &= \frac{114}{256}. \end{aligned}$$

Problem 4

Find the cardinality (=size=number of elements) of the following set: the set of odd numbers with distinct digits between 1000 and 10000.

Solution

There are four positions to fill: the units, the tens, the hundreds, and the thousands.

a) There are 5 options for the units: $\{1, 3, 5, 7, 9\}$.

b) For the thousands position, we cannot use 0 and we cannot use the number that was used in units position. So we are left with 8 choices for the thousands position.

c) For the hundreds position, we cannot use the numbers used in the thousands or the units positions, so again we are left with 8 choices.

d) For the tens, we cannot use any of the previously used digits. So we have 7 choices.

Hence, we have $5 \times 8 \times 8 \times 7$ choices.

Problem 5

How many positive integers are there that are less than 1000 and have exactly one 5?

Solution

The 5 can be any of the following positions: units, tens, or hundreds. So we have 3 options. The two remaining positions can be filled in 9×9 ways. So there are 243 such numbers.