

Problem Set 1

Reading: Axioms of Probability

Quiz Date: Wednesday, June 13

Problem 1

State the event axioms and the probability axioms.

Solution

See page 8 of notes.

Problem 2

Consider a sample space Ω and a corresponding set of events \mathcal{F} . Assume A , B , and C are events. That is, $A, B, C \in \mathcal{F}$. Prove the following statements.

- $A \cap B \in \mathcal{F}$
- $P(A) \leq 1$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Solution

See property e.5 on page 8 and properties p.7 and p.8 on page 9 of the notes.

Problem 3

Suppose a fair die is rolled twice.

- Find Ω .
- Let A be the event that the first roll is smaller than 3. Find A and $P(A)$.
- Let B be the event that the sum of the rolls equals 4. Find B and $P(B)$.

Solution

- We can represent the outcomes of the experiments using an ordered pair (a, b) where a is the number showing in the first roll and b the number showing in the second roll. We can express Ω by writing its elements:

$$\Omega = \{(1, 2), (1, 3), \dots, (1, 6), (2, 2), (2, 3), \dots, (2, 6), \dots, (6, 6)\}$$

but a shorter and less ambiguous way is describing the elements:

$$\Omega = \{(a, b) : a \in \mathbb{N}, b \in \mathbb{N}, 1 \leq a \leq 6, 1 \leq b \leq 6\}.$$

When Ω is a finite set, most often, we take \mathcal{F} to be the set of all subsets of Ω .

b) Similar to the previous part, we may write

$$A = \{(1, 2), (1, 3), \dots, (1, 6), (2, 2), (2, 3), \dots, (2, 6)\}$$

or

$$A = \{(a, b) : a \in \mathbb{N}, b \in \mathbb{N}, 1 \leq a \leq 2, 1 \leq b \leq 6\}.$$

Since the die is fair, all possible outcomes have the same probability. So

$$P(A) = \frac{|A|}{|\Omega|} = \frac{12}{36} = \frac{2}{6} = \frac{1}{3}.$$

c) $B = \{(1, 3), (2, 2), (3, 1)\}$ and $P(B) = |B|/|\Omega| = 3/36 = 1/12$.

Problem 4

Consider events A and B defined on a sample space. Suppose the probability that at least one of the two events occurs is 0.6 and the probability that at least one of the events does not occur is 0.8. What is the probability that exactly one of the two events occurs?

Solution

From the statement of the problem, we have $P(A \cup B) = 0.6$ and $P(A^c \cup B^c) = 0.8$. Note that $P(A \cup B) = P(A \cap B) + P(A \cap B^c) + P(A^c \cap B)$ and $P(A^c \cup B^c) = P(A^c \cap B^c) + P(A^c \cap B) + P(A \cap B^c)$. So

$$\begin{aligned} P(A \cap B) + P(A \cap B^c) + P(A^c \cap B) &= 0.6, \\ P(A^c \cap B^c) + P(A^c \cap B) + P(A \cap B^c) &= 0.8. \end{aligned}$$

Summing up the above equations, we obtain

$$P(A \cap B) + P(A \cap B^c) + P(A^c \cap B) + P(A^c \cap B^c) + P(A^c \cap B) + P(A \cap B^c) = 1.4$$

and since $P(A \cap B) + P(A \cap B^c) + P(A^c \cap B) + P(A^c \cap B^c) = P(\Omega) = 1$, we get

$$P(A^c \cap B) + P(A \cap B^c) = 0.4.$$