ECE 313

Midterm Exam II

July 10, 2012

Exam Time : 100 mins

Problem 1

(9+8 pts)

Consider the following pdf for a random variable X.

$$f_X(u) = \begin{cases} \frac{24}{u^4}, & u \geq a \\ 0, & u < a \end{cases}$$

a) Find the value of a such that f_X is a valid pdf.

b) Find $E[X]$.

Solution

a) The value of a should be chosen such that $\int_{-\infty}^{\infty} f_X(u)\,du = 1$. We have

$$\int_{-\infty}^{\infty} f_X(u)\,du = \int_a^{\infty} \frac{24}{u^4} du = 8\left(-\frac{1}{u^3}\right)_a^{\infty} = \frac{8}{a^3},$$

so we let $a = 2$.

b)

$$E[X] = \int_2^{\infty} u\frac{24}{u^4} du = \int_2^{\infty} \frac{24}{u^3} du = 12\left(-\frac{1}{u^2}\right)_2^{\infty} = \frac{12}{4} = 3.$$

Problem 2

(8+2+8 pts)

A sports competition is held among 16 European national teams, among them are Spain, Germany, Portugal, and Italy. The teams are to be divided into four groups, each with four teams. Suppose teams are grouped randomly, with all possibilities being equally likely.

a) What is the probability that Spain and Italy are in the same group?

b) Which is larger? Indicate your answer by circling the appropriate item. (You can either guess or use the result of part c)

i) the probability that Spain and Italy are in the same group, or

ii) the conditional probability that Spain and Italy are in the same group given that Portugal and Germany are in the same group.

c) What is the conditional probability that Spain and Italy are in the same group given that Portugal and Germany are in the same group?

Solution

a) Spain is in some group. That group has three other members. So the desired probability is $3/15$ where 15 is the number of remaining positions. Hence, the final answer is $3/15 = 1/5$.

b) ii is larger. See part c.

c) There are two disjoint cases: Either

- all four teams are in the same group: The probalitiy corresponding to this case is

$$\frac{2}{14} \cdot \frac{1}{13}$$

- Spain and Italy in one group and Portugal and Germany in another group. The probability corresponding to this case is

$$\frac{12}{14} \cdot \frac{3}{13}$$

The total probality is thus $\frac{2+12\cdot 3}{14\cdot 13} = \frac{38}{182} = \frac{19}{91}$. Note that $\frac{19}{91} > \frac{1}{5}$.

Problem 3

(9 pts)

We are told of the following experiment. First a fair coin is flipped n times and it is noted that Heads is observed 3 times. Next, the same coin is flipped n times once again and this time Heads is observed 6 times. The value of n is unknown to us. With the information given, find the Maximum Likelihood estimate of n. (You *may* find this useful: $\sqrt{12} \simeq 3.46$).

Solution

The number of Heads in the first round, X_1, is a Binomial with parameters $(n, 1/2)$. For the second round, the number of Heads, X_2, is again a Binomial with parameters $(n, 1/2)$. These two random variables are independent. So the likelihood function is

$$l(n) = P(X_1 = 3, X_2 = 6) = P(X_1 = 3) P(X_2 = 6) = \binom{n}{3}\left(\frac{1}{2}\right)^n \binom{n}{6}\left(\frac{1}{2}\right)^n.$$

By definition of ML estimate, $\hat{n}_{ML}$ is the value of n that maximizes $l(n)$. To find $\hat{n}_{ML}$, we find

$$\frac{l(n)}{l(n-1)} = \frac{\binom{n}{3}\left(\frac{1}{2}\right)\binom{n}{6}\left(\frac{1}{2}\right)}{\binom{n-1}{3}\binom{n-1}{6}} = \frac{n^2}{4(n^2 - 9n + 18)}$$

Comparing the ration with 1 yields

$$\frac{n^2}{4\left(n^2-9n+18\right)} < 1 \iff n^2 < 4n^2 - 36n + 72 \iff 3\left(n^2 - 12n + 24\right) > 0$$
$$\iff n^2 - 12n + 24 > 0 \iff (n-6)^2 > 12 \iff 6 - \sqrt{12} < n < 6 + \sqrt{12}.$$

We know that $n \geq 6$. In the region of $n \geq 6$, $l(n)$ is increasing for $n < 6 + \sqrt{12}$. Thus $\hat{n}_{ML} = \lfloor 6 + \sqrt{12} \rfloor = 9$.

Problem 4

(7+6+6 pts)

The random variable X is defined as follows:

A biased coin, with $P(\text{Heads}) = 1/3$, is tossed. If Heads shows, we let $X = 0$. If Tails shows, we randomly and uniformly choose the value of X from the interval $[0, 1]$.

a) Sketch the CDF of X. Clearly mark important points and values. Use full and empty circles to indicate the value of the CDF at discontinuities.

b) Find $E[X]$. Hint: You can use the law of total probability for expectation.

c) Find $\text{Var}[X]$.

Solution

a)

$$F_X(c) = \begin{cases} 0, & c < 0 \\ \frac{2}{3}c + \frac{1}{3}, & 0 \leq c \leq 1 \\ 1, & c > 1 \end{cases}$$

Discontinuity at $c = 0$.

b) Let H denote the event that the coin shows Heads and T denote the event that the coin shows Tails.

$$E[X] = E[X|T]\,P(T) + E[X|H]\,P(H) = \frac{1}{2} \cdot \frac{2}{3} + 0 \cdot \frac{1}{3} = \frac{1}{3}.$$

c) To find the variance, we find $E\left[X^2\right]$.

$$E\left[X^2\right] = E\left[X^2|T\right]P(T) + E\left[X^2|H\right]P(H) = \frac{2}{3}\int_0^1 u^2 du + 0 \cdot \frac{1}{3} = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$$

So $\text{Var}[X] = \frac{2}{9} - \left(\frac{1}{3}\right)^2 = \frac{1}{9}$.

Problem 5

(10+9 pts)

We are given two boxes. In each box there are 5 black marbles and 4 white marbles. We pick a marble at random from the first box and put it in the second box. Then, we pick a marble randomly from the second box.

Let B_i, for $i = 1, 2$, be the event that the marble picked from the ith box is black. For example B_2 is the event that the marble picked from the second box is black. Similarly, define W_i as the event that the marble picked from the ith box is white. Find

a) $P(B_2)$.

b) $P(W_1|B_2)$.

Solution

a)

$$P(B_2) = P(B_2|W_1)\,P(W_1) + P(B_2|B_1)\,P(B_1) = \frac{5}{10}\cdot\frac{4}{9} + \frac{6}{10}\cdot\frac{5}{9} = \frac{2}{9} + \frac{3}{9} = \frac{5}{9}.$$

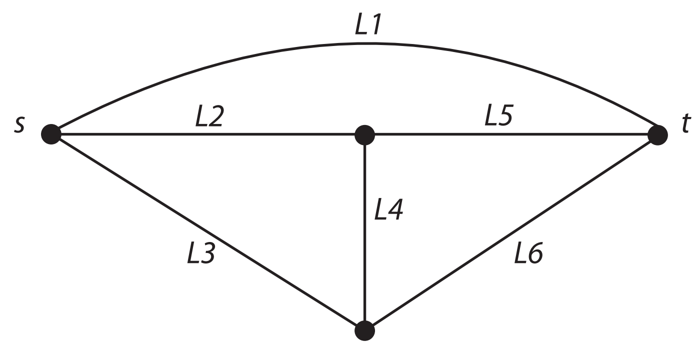
b)

$$P(W_1|B_2) = \frac{P(B_2, W_1)}{P(B_2)} = \frac{P(B_2|W_1)\,P(W_1)}{P(B_2)} = \frac{2/9}{5/9} = \frac{2}{5}.$$

Problem 6

(9+9 pts)

In the following network, all links L_i have capacity $C_i = 1$. Each link L_i may fail independently of others with probability p_i. The source is node s and the destination is node t. Let the capacity of the network be denoted by X.



a) Find $P(X = 3)$.

b) Assume that $p_4 = 1$. This is equivalent to assuming that L_4 is not in the network. Use union bound to obtain an upper-bound on the probability of network outage (network failure).

Solution

a) Let F_i be the event that link L_i fails. For $X = 3$, all links must work except L_4. So

$$\begin{aligned} P(X=3) &= P\left(F_1^c F_2^c F_3^c F_5^c F_6^c\right) \\ &= (1-p_1)(1-p_2)(1-p_3)(1-p_5)(1-p_6). \end{aligned}$$

b) Let F be the event that the network fails.

$$\begin{aligned} P(F) &= P\left(F_1 (F_2 \cup F_5)(F_3 \cup F_6)\right) \\ &= P(F_1)\,P(F_2 \cup F_5)\,P(F_3 \cup F_6) \\ &\le P(F_1)\left(P(F_2) + P(F_5)\right)\left(P(F_3) + P(F_6)\right) \\ &= p_1 (p_2 + p_5)(p_3 + p_6). \end{aligned}$$