ECE 313
Midterm Exam I

June 26, 2012

Exam Time : 100 mins

Problem 1

(30 points)

True or False? A statement is True if and only if it is always true. Each correct choice counts +3 points, whereas an incorrect choice counts -1 point.

a) True ☐ False ☐ For two events A and B, $P(A \cup B) = P(A) + P(B)$.

b) True ☐ False ☐ If A, B, and C are independent events, then A is independent of $B \backslash C$. (For two sets S and T, $S \backslash T$ is defined is $S \cap T^c$)

c) True ☐ False ☐ For random variables X and Y, we have $E\left[X + Y^2\right] = E[X] + (E[Y])^2$.

d) True ☐ False ☐ The following is a valid pmf for a random variable X.

$$p_X(i) = (e-1)e^{-i} \qquad \text{for } i = 1, 2, \cdots$$

e) True ☐ False ☐ For random variables X and Y, we have $\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$.

f) True ☐ False ☐ Let A and B be events with probabilities $P(A) = 3/4$ and $P(B) = 2/3$. It is possible to have $P(A \cap B) = 1/6$.

g) True ☐ False ☐ For sets A, B, and C, $A \backslash (B \cap C^c) = (A \backslash B) \cup (A \backslash C^c)$.

h) True ☐ False ☐ Let X be a geometric random variable with parameter $0 < p < 1$. The pmf $p_X(k)$ obtains its maximum at $\lfloor 1/p \rfloor$.

i) True ☐ False ☐ A woman has two children. The conditional probability that both her children are boys given that at least one is a boy equals $1/3$.

j) True ☐ False ☐ Two teams A and B play a best-of-five series of games. Assume that games are independent, ties are not possible, and A wins each game with probability $2/3$. The series ends once one of the teams has won three games. The probability that A wins the series in four games is $32/81$.

Problem 2

(18=9+9 points)

A standard deck of 52 cards is randomly divided into four piles, each containing 13 cards. (A standard pack has 13 spades, 13 hearts, 13 clubs, and 13 diamonds.) Find

a) the probability that "the ace of spades and the ace of hearts are in the same pile".

b) the conditional probability that "the ace of spades and the ace of hearts are in the same pile given that two of the piles do not contain any aces".

Problem 3

(18 points)

A die is rolled three times. Let X denote the second largest number. For example, if the outcome is $(2, 3, 1)$, then $X = 2$; if the outcome is $(3, 3, 5)$, then $X = 3$; if the outcome is $(4, 4, 4)$ then $X = 4$. Find the pmf of X.

Problem 4

(16=8+8 points)

UIUC has approximately 40000 students; 47% female and 53% male. The college of engineering of UIUC has approximately 10000 students, 20% of whom are female. A student is chosen at random.

a) What is the conditional probability that the student is male given that the student is not in the college of engineering?

b) What is the conditional probability that the student is in the college of engineering given that the student is female?

Problem 5

(18 points)

Consider repeated independent rolls of a fair die and let X denote the number of rolls required to observe all even numbers, that is, 2, 4, and 6.

a) What is the conditional probability that $X = 7$ given that the first four rolls are 3,2,5,6.

b) Find $E[X]$.

c) Find $\mathrm{Var}[X]$.