

# A SHARP ESTIMATE FOR THE HILBERT TRANSFORM ALONG FINITE ORDER LACUNARY SETS OF DIRECTIONS

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ABSTRACT. Let  $D$  be a nonnegative integer and  $\Theta \subset S^1$  be a lacunary set of directions of order  $D$ . We show that the  $L^p$  norms,  $1 < p < \infty$ , of the maximal directional Hilbert transform in the plane

$$H_{\Theta}f(x) := \sup_{v \in \Theta} \left| \text{p.v.} \int_{\mathbb{R}} f(x + tv) \frac{dt}{t} \right|, \quad x \in \mathbb{R}^2,$$

are comparable to  $(\log \#\Theta)^{\frac{1}{2}}$ . For vector fields  $\mathbf{v}_D$  with range in a lacunary set of of order  $D$  and generated using suitable combinations of truncations of Lipschitz functions, we prove that the truncated Hilbert transform along the vector field  $\mathbf{v}_D$ ,

$$H_{\mathbf{v}_D,1}f(x) := \text{p.v.} \int_{|t| \leq 1} f(x + t\mathbf{v}_D(x)) \frac{dt}{t},$$

is  $L^p$ -bounded for all  $1 < p < \infty$ . These results extend previous bounds of the first author with Demeter, and of Guo and Thiele.

## 1. INTRODUCTION

Our subject is Hilbert transforms, and more general singular integrals and maximal averages along directions in the plane. The simplest example of such an operator is described by fixing a nonzero vector  $v \in \mathbb{R}^2$  and considering the multiplier

$$(1.1) \quad H_v^+ f(x) := \int_{\mathbb{R}^2} \widehat{f}(\xi) \mathbf{1}_{[0,\infty)}(\xi \cdot v) e^{ix \cdot \xi} d\xi, \quad x \in \mathbb{R}^2,$$

acting initially on Schwartz functions  $f$  defined on  $\mathbb{R}^2$ . Up to a linear combination with the identity operator, this operator coincides with the Hilbert transform along the direction  $v$

$$H_v f(x) := \text{p.v.} \int_{\mathbb{R}} f(x + tv) \frac{dt}{t}.$$

Notice that  $H_v$  is dilation invariant and thus we can always replace  $v$  by its projection on  $S^1$ , which we identify with  $v$  when  $v$  is acting as a parameter for the definition of  $H_v$ .

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The companion maximal operator is the maximal average of a function  $f$  in the direction given by  $v$

$$M_v f(x) := \sup_{\varepsilon > 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |f(x + tv)| dt, \quad x \in \mathbb{R}^2.$$

Such operators arise for example if one applies the method of rotations to singular integrals in  $\mathbb{R}^2$  which are given by  $-2$ -homogeneous kernels with mean zero on  $S^1$ . For a fixed direction  $v \in S^1$ , one-dimensional theory implies that  $H_v, M_v$  are bounded operators on  $L^p(\mathbb{R}^2)$ ,  $1 < p < \infty$ , with bounds independent of  $v$ . Of course the maximal operator  $M_v$  is also trivially bounded on  $L^\infty(\mathbb{R}^2)$ . See for example [11, §4.3] for a relevant discussion.

Things become more interesting when one seeks for bounds for the corresponding maximal versions of these operators along sets of directions. To make this more precise, let  $\Theta \subset S^1$  and consider the operators

$$M_\Theta f(x) := \sup_{v \in \Theta} M_v f(x), \quad H_\Theta f(x) := \sup_{v \in \Theta} |H_v f(x)|, \quad x \in \mathbb{R}^2.$$

We naturally ask under which conditions on  $\Theta$  these operators are bounded on  $L^p(\mathbb{R}^2)$ , at least for some  $p \in (1, \infty)$ . In order to discuss the answer we need the definition of lacunary sets.

*Lacunary sets.* Let  $\lambda > 1$  be a fixed parameter. A 0-lacunary set is a single point set  $\Theta = \{v_\infty\}$  with  $v_\infty \in S^1$ . A 1-lacunary set  $\Theta \subset S^1$  with limit  $v_\infty \in S^1$  is a countable set with the property

$$\forall v \in \Theta, \quad \inf_{\substack{v' \in \Theta \\ v' \neq v}} \text{dist}(v, v') \geq \lambda \text{dist}(v, v_\infty).$$

Here by  $\text{dist}$  we mean geodesic distance on  $S^1$ . Observe that a 1-lacunary set with limit  $v_\infty$  can be written as the disjoint union of two lacunary sequences converging to  $v_\infty$ , one clockwise towards  $v_\infty$ , one counterclockwise towards  $v_\infty$ .

Now for a positive integer  $D > 1$  we define a  $D$ -lacunary set inductively. Assuming that we have defined the  $(D-1)$  case, a set  $\Theta \subset S^1$  will be called a  $D$ -lacunary set if  $\Theta$  can be written as a union of 1-lacunary sets with limits in a  $(D-1)$ -lacunary set  $\Theta'$ . In this case, we say that  $\Theta$  is a successor of  $\Theta'$ . It is of some importance to note here that the parameter  $\lambda$  should remain fixed throughout the definition of every order of lacunarity. Note that, with this definition, every  $D$ -lacunary set  $\Theta$  is associated with a single element  $v_\infty$  that will be called *the root* of  $\Theta$ .

With the definition of a  $D$ -lacunary set in hand, we now return to the questions of boundedness of  $M_\Theta$  and  $H_\Theta$ . If  $\Theta$  is a  $D$ -lacunary set then  $M_v$  is bounded on  $L^p(\mathbb{R}^2)$  for all  $1 < p < \infty$ . For  $D = 1$  this is the main result of [20]. This result was extended to lacunary sets of finite order in [24]. The converse implication is a striking result of Bateman, [1]: if  $M_v$  is bounded on  $L^p(\mathbb{R}^2)$  for some  $p \in (1, \infty)$  then  $\Theta$  is a  $D$ -lacunary set for some nonnegative integer  $D$ . In [17], Katz proved the quantitative sharp bounds for arbitrary finite sets of directions  $\Theta$

$$\|M_\Theta\|_{L^2 \rightarrow L^2} \lesssim \log \#\Theta, \quad \|M_\Theta\|_{L^2 \rightarrow L^{2,\infty}} \lesssim (\log \#\Theta)^{\frac{1}{2}}, \quad \|M_\Theta\|_{L^p \rightarrow L^p} \lesssim (\log \#\Theta)^{\frac{1}{p}}, \quad p > 2.$$

Note that the result from [24] implies that  $\|M_\Theta\|_{L^p \rightarrow L^p} \lesssim_D 1$  whenever  $\Theta$  is a  $D$ -lacunary set of directions.

For the Hilbert transform  $H_\Theta$ , Karagulyan showed in [16] that for all set  $\Theta \subset S^1$  we have

$$\|H_\Theta\|_{L^2(\mathbb{R}^2)} \gtrsim (\log \#\Theta)^{\frac{1}{2}}.$$

In particular, the Hilbert transform  $H_\Theta$  is unbounded whenever  $\Theta$  is an infinite set of directions.

Concerning upper bounds for  $H_{\Theta}$ , keeping in mind the above result of Karagulyan, we only consider  $\Theta \subset S^1$  with  $\#\Theta < \infty$ . Then the main result is due to the first author and Demeter [9]: if  $\Theta$  is any finite set of directions then  $\|H_{\Theta}\|_{L^p \rightarrow L^p} \lesssim_p \log \#\Theta$  for all  $2 \leq p < \infty$ . The case  $p = 2$  of these norm bounds was previously treated in [8]. In [9] it is also shown that under additional structural assumptions on  $\Theta$ , including for instance the equi-spaced case  $\Theta = \{e^{2\pi i \frac{k}{N}} : k = 1, \dots, N\}$ , the quantitative estimate improves to  $\|H_{\Theta}\|_{L^p \rightarrow L^p} \lesssim_{p,\varepsilon} (\log \#\Theta)^{\frac{1}{2} + \varepsilon}$  for  $2 < p < 2 + \varepsilon$  and  $\varepsilon > 0$  sufficiently small: the proof uses product-BMO type phase plane analysis.

In the same paper [9] the authors prove that if  $\Theta$  is a 1-lacunary set then the upper bound improves to  $\|H_{\Theta}\|_{L^p \rightarrow L^p} \lesssim_p (\log \#\Theta)^{\frac{1}{2}}$  for all  $1 < p < \infty$ . The first main result of this paper is the  $D$ -lacunary version of the upper bounds for  $H_{\Theta}$ .

**Theorem 1.** *Let  $\Theta \subset S^1$  be a  $D$ -lacunary set of directions. The maximal directional Hilbert transform  $H_{\Theta}f(x) := \sup_{v \in \Theta} |H_v f(x)|$  obeys the bounds*

$$(1.2) \quad c(\log \#\Theta)^{\frac{1}{2}} \leq \|H_{\Theta}\|_{L^p(\mathbb{R}^2)} \leq C(\log \#\Theta)^{\frac{1}{2}}, \quad 1 < p < \infty,$$

with constants  $c, C > 0$  depending only on  $D, p$ .

The maximally truncated Hilbert transform in the direction  $v \in S^1$  is

$$H_v^* f(x) := \sup_{\varepsilon > 0} \left| \int_{|t| > \varepsilon} f(x - tv) \frac{dt}{t} \right|$$

and then we set  $H_{\Theta}^* f(x) := \sup_{v \in \Theta} H_v^* f(x)$ . The previous theorem together with the one-dimensional Cotlar inequality for the maximal truncations of the Hilbert transform, see for example [11, Lemma 3.5],

$$H_v^* f(x) \leq M_v(H_{v_i} f)(x) + CM_v f(x),$$

implies the corresponding estimate for  $H_{\Theta}^*$

$$H_{\Theta}^* f(x) \leq M_{\Theta}(H_{\Theta} f)(x) + CM_{\Theta} f(x).$$

This together with the boundedness of  $M_{\Theta}$  on  $L^p(\mathbb{R}^2)$  for all  $p > 1$  immediately imply the following corollary for the maximal truncations of  $H_{\Theta}$ .

**Corollary 1.1.** *Let  $\Theta \subset S^1$  be a  $D$ -lacunary set of directions. We have*

$$\|H_{\Theta}^*\|_{L^p(\mathbb{R}^2)} \leq C(\log \#\Theta)^{\frac{1}{2}}, \quad 1 < p < \infty,$$

where  $C > 0$  depends only on  $D, p$ .

The results and setup described above naturally connect to the Hilbert transform along vector fields. To make this precise, let  $\Theta$  be a set of directions and consider the operator

$$H_{\Theta} f(x) := \sup_{v \in \Theta} \left| \text{p.v.} \int_{\mathbb{R}} f(x + tv) \frac{dt}{t} \right|.$$

Then there exists a measurable function  $\mathbf{v} : \mathbb{R}^2 \rightarrow \Theta$  such that, up to a sign,  $H_{\Theta}$  is pointwise equivalent to

$$f \mapsto \text{p.v.} \int_{\mathbb{R}} f(x + t\mathbf{v}(x)) \frac{dt}{t}.$$

As described in [19], there is a smooth vector field  $\mathbf{v}$  such that the untruncated version considered above fails to be  $L^2$ -bounded even after precomposition with a smooth restriction to a frequency annulus. Such obstruction, caused by the large scales of the kernel, is obviated naturally by considering instead the operator

$$H_{\mathbf{v},\varepsilon}f(x) := \text{p.v.} \int_{|t|\leq\varepsilon} f(x + t\mathbf{v}(x)) \frac{dt}{t}$$

where  $\mathbf{v} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a sufficiently smooth vector field. The truncation at scale  $\varepsilon$  should be chosen in inverse relation to some modulus of smoothness of  $\mathbf{v}$ . Similarly, one is led to consider the truncated maximal average of a Schwartz function  $f$  along the vector field  $\mathbf{v}$

$$M_{\mathbf{v},\varepsilon}f(x) := \sup_{0 < s \leq \varepsilon} \frac{1}{2s} \int_{|t|\leq s} |f(x + t\mathbf{v}(x))| dt.$$

The study of this operator is more naturally motivated by the question of differentiation of functions along vector fields, and then the truncation to small scales is immediately justified.

In the context described above, Zygmund conjectured the weak- $L^2$  boundedness of the maximal function  $M_{\mathbf{v},\varepsilon}$  whenever  $\mathbf{v}$  is a Lipschitz vector field and  $\varepsilon^{-1} \simeq \|\mathbf{v}\|_{\text{LIP}}$ . The analogous boundedness property for the Hilbert transform  $H_{\mathbf{v},\varepsilon}$  under the same assumptions on  $\mathbf{v}$  and  $\varepsilon$  was later conjectured by Stein [25]. These conjectures have been answered in the affirmative in the case that  $\mathbf{v}$  is an *analytic* vector field: by Bourgain, [4], for the maximal operator and by Stein and Street, [26], for the Hilbert transform. These results were later recovered and extended by Lacey and Li to  $C^{1+\eta}$ -vector fields,  $\eta > 0$ , satisfying an additional geometric condition which holds when  $\mathbf{v}$  is analytic; see [18, 19]. For further refinements and additional results we point the interested reader to [2, 3, 13, 14, 15] and to the references therein.

The second main result of the paper, Theorem 2 below, extends the class of vector fields along which the truncated Hilbert transform admits  $L^p$ -bounds to include those with  $D$ -lacunary range and satisfying an additional assumption of Lipschitz flavor. Our result extends [15, Theorem 1.2] by Guo and Thiele to lacunary orders  $D > 1$ . In the statement, and in what follows, we denote by  $\lfloor x \rfloor$  the largest integer which is less than or equal to  $x$ .

**Theorem 2.** *Let  $D$  be a positive integer. For  $j = 1, \dots, D$  let  $\lambda_j : \mathbb{R}^2 \rightarrow (0, 1]$  be Lipschitz functions with  $\|\lambda_j\|_{\text{LIP}} \leq 1$  and with the additional property that*

$$\lambda_j(x) \leq 2^{-5} \lambda_{j-1}(x), \quad \forall x \in \mathbb{R}^2, \quad j = 2, \dots, D.$$

Define  $\mathbf{v}_D : \mathbb{R}^2 \rightarrow S^1$  by

$$(1.3) \quad \mathbf{v}_D(x) := \prod_{j=1}^D e^{2\pi i 2^{\lfloor \log_2 \lambda_j(x) \rfloor}}.$$

Then the truncated Hilbert transform  $H_{\mathbf{v}_D,1}$  satisfies the norm inequality

$$(1.4) \quad \|H_{\mathbf{v}_D,1}f\|_{L^p(\mathbb{R}^2)} \leq C \|f\|_{L^p(\mathbb{R}^2)}, \quad 1 < p < \infty,$$

with constant  $C$  depending on  $p, D$  only.

**Remark 1.2.** While we chose a specific  $D$ -lacunary sequence as the range of  $\mathbf{v}_D$  in Theorem 2, namely

$$\mathbf{v}_D(\mathbb{R}^2) \subset \left\{ \prod_{j=1}^D e^{2\pi i 2^{-k_j}} : (k_1, \dots, k_D) \in \mathbb{N}^D, k_1 < k_2 < \dots < k_D \right\},$$

the statement and proof of our theorem, as well as that of [15], can be suitably modified to fit a vector field whose range is any given  $D$ -lacunary set.

**Remark 1.3.** The same Lipschitz-lacunary linearization of [15], namely (1.3) with  $D = 1$ , was employed by Saari and Thiele in the context of  $L^p$ -bounds for maximal multipliers with hyperbolic cross-type singularity [23]. In light of Theorem 2 it seems worthwhile to investigate whether the results of [23] hold with the more general linearization (1.3).

**Remark 1.4.** In both Theorems 1 and 2, the inequality for the Hilbert transform along  $\mathbf{v}$  is reduced to a Littlewood-Paley type square function estimate, explicitly in the first case (see Proposition 2.2, using a version of the Chang-Wilson-Wolff inequality) or implicitly in the second, relying upon the Lipschitz character of  $\mathbf{v}$ . This reduction is a common theme in the literature [3, 8, 9, 18]; on the other hand, the proofs of boundedness for such square functions rely upon certain  $L^p$  estimates for maximal functions oriented along the direction of the vector fields. Our results exhaust the cases where, due to the finite order lacunarity of the range, such maximal functions have a full range of  $L^p$ -bounds without any further assumption on the vector field (see (2.1) below). It was shown by Parcet and Rogers in [21] that suitably defined higher order lacunary sets in general dimension give rise to  $L^p$ -bounded directional maximal functions: in light of this result, it is natural to ask whether our Theorems 1 and 2 extend to the higher dimensional case.

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## 2. AUXILIARY RESULTS

In this section, we collect some tools which will be employed within the proofs of the main results. We begin with an important section that concerns the notations and assumptions used in this paper.

**2.1. Some conventions concerning lacunary sets of directions.** Let  $\Theta \subset S^1$  be a  $D$ -lacunary set of directions. We identify a direction  $v \in \Theta$  with its argument, namely  $v := e^{2\pi i \theta_v}$ . Then  $v^\perp = e^{2\pi i(\theta_v + 1/4)} =: e^{2\pi i \theta_{v^\perp}}$ . By a slight abuse of notation we will also write  $\theta^\perp := \theta + \frac{1}{4}$  so that  $\theta_{v^\perp} = \theta_v^\perp$ . Here we remember that a  $D$ -lacunary set  $\Theta$  is a successor of a  $(D-1)$ -lacunary set  $'\Theta$  and can be written as a union of 1-lacunary sets with limits in  $'\Theta$ . In general, there will be two sequences of elements of  $\Theta$  converging to a given  $u \in '\Theta$ , one in clockwise order, and another in counterclockwise order. However, by symmetry it suffices to consider only the clockwise ordered sequences. Thus, all lacunary sets in this paper will be in clockwise

order with respect to their limits and a 1-lacunary set can be identified with an ordered set  $\Theta = \{v_1, \dots, v_j, \dots\}$ .

A further convention that we will use in this paper is that the  $D$ -lacunary sets we consider will lie in the first quadrant so that for each  $v \in \Theta$  we have  $0 \leq \theta_v \leq 1/4$ . Without loss of generality we assume throughout the paper that the root  $v_\infty$  of  $\Theta$  is  $v_\infty = (1, 0) = e^{2\pi i 0}$ .

A 1-lacunary set with lacunarity constant  $\lambda$  can be split into roughly  $\log(\lambda'/\lambda)$  pairwise disjoint 1-lacunary sets with the same limit and lacunarity constant  $\lambda'$ . Inductively, a  $D$ -lacunary set with constant  $\lambda$  can be split into roughly  $[\log(\lambda'/\lambda)]^D$  pairwise disjoint  $D$ -lacunary sets with constant  $\lambda'$ . Given a  $D$ -lacunary set  $\Theta = \cup_\ell \Theta_\ell$ , the maximal Hilbert transform  $H_\Theta$  can be trivially controlled from above by  $\sum_\ell H_{\Theta_\ell}$ . As we can tolerate constants depending on  $\lambda$  and  $D$ , we will, without particular mention, assume that the lacunarity constant of a given  $D$ -lacunary set is sufficiently large, whenever needed. Note that this splitting will only introduce constants depending on  $\lambda$  and  $D$ .

**2.2. A representation of the Hilbert transform in terms of cone projections.** We continue with the introduction of a notation that we will consistently use throughout the paper. Given a  $D$ -lacunary set  $\Theta = \{v_j\}_{j \geq 1}$  we always write  $'\Theta = \{u_\tau\}_{\tau \geq 1}$  if  $\Theta$  is a successor of  $'\Theta$ . Then  $\Theta = \cup_{\tau \geq 1} \Theta_\tau$  where each set  $\Theta_\tau$  is a 1-lacunary set with limit  $u_\tau$ . As each  $\Theta_\tau$  is 1-lacunary, and thus ordered, we now agree to write  $\Theta_\tau = \{v_{j_{\tau-1}}, v_{j_{\tau-1}+1}, \dots, v_j, \dots\}$  where  $v_{j_{\tau-1}} := u_{\tau-1}$  for  $\tau \geq 2$ . For  $\tau = 1$  we just use the convention  $u_0 := v_1$  and the same definition makes sense.

Define the cones

$$C_j := \{\xi \in \mathbb{R}^2 : \xi \cdot v_j \geq 0, \xi \cdot v_{j+1} < 0, \xi_1 < 0, \xi_2 > 0\},$$

and

$$\tilde{C}_\tau := \{\xi \in \mathbb{R}^2 : \xi \cdot u_\tau \geq 0, \xi \cdot u_{\tau+1} < 0, \xi_1 < 0, \xi_2 > 0\},$$

which should be understood as regions in the frequency plane; see also Figure 1 below. Then

$$\{\xi \in \mathbb{R}^2 : \xi_1 < 0, \xi_2 > 0, \xi \cdot v_1 \geq 0\} = \bigcup_{j \geq 1} C_j = \bigcup_{\tau \geq 1} \bigcup_{j_{\tau-1} \leq j < j_\tau} C_j = \bigcup_{\tau \geq 1} \tilde{C}_{\tau-1}.$$

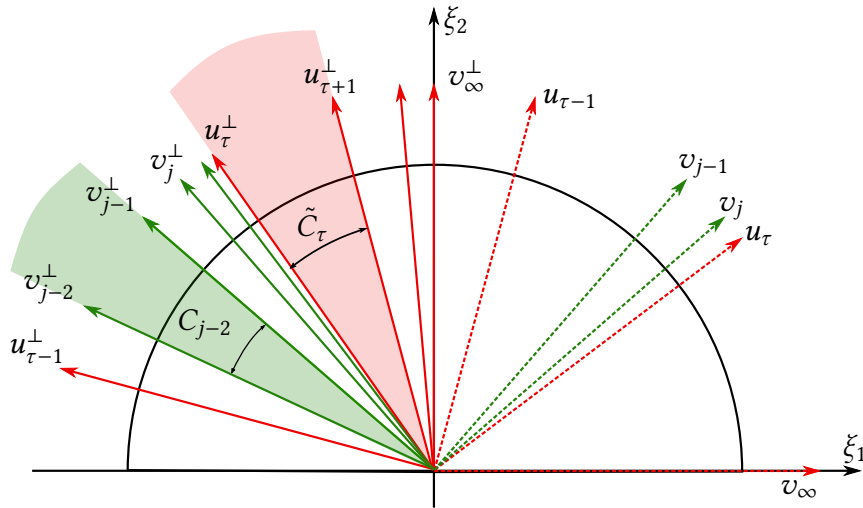


FIGURE 1. Lacunary sets and cones

Given a  $D$ -lacunary set  $\Theta$  which is successor to  $'\Theta$  we will write  $R_j$  for the Fourier restrictions  $\widehat{R_j f} := \widehat{f} \mathbf{1}_{C_j}$  and similarly for  $\tilde{R}_\tau$ . We have actually proved the following representation for the operator  $H_\Theta^+$ .

**Lemma 2.1.** *Let  $\Theta = \{v_j\}_{j \geq 1}$  be a  $D$ -lacunary set of directions following the assumptions of this paragraph, and assume that  $\Theta$  is successor to  $'\Theta = \{u_\tau\}_{\tau \geq 1}$  and write  $\Theta = \cup_{\tau \geq 1} \Theta_\tau$  as above. Then for all  $g \in \mathcal{S}(\mathbb{R}^2)$  such that  $\text{supp } \widehat{g} \subseteq \{\xi \in \mathbb{R}^2 : \xi_1 < 0, \xi_2 > 0\}$ , we have*

$$H_\Theta^+ g(x) = \sup_j \left| \sum_{\tau \geq j} R_\tau g(x) \right|.$$

Furthermore, for  $g$  as above we have the recurrence estimate

$$H_\Theta^+ g(x) \leq H_{\Theta'}^+ f + \sup_{\tau \geq 1} H_{\Theta_\tau}^+ (\tilde{R}_{\tau-1} g).$$

*Proof.* The first identity of the lemma follows immediately from the the discussion before its statement. To see the second estimate of the lemma, fix some  $j$  and suppose that  $j_{\tau-1} < j \leq j_\tau$  for some  $\tau \geq 1$ . Using the representation formula of the lemma we easily obtain

$$\begin{aligned} H_{v_j}^+ (g) &= \sum_{\ell \geq j} R_\ell g = \sum_{j \leq \ell < j_\tau} R_\ell g + \sum_{\ell \geq j_\tau} R_\ell g = \sum_{j \leq \ell < j_\tau} R_\ell (\tilde{R}_{\tau-1} g) + \sum_{\ell \geq j_\tau} \tilde{R}_\ell (g) \\ &= \sum_{j \leq \ell < j_\tau} R_\ell (\tilde{R}_{\tau-1} g) + H_{u_\tau}^+ g. \end{aligned}$$

Thus we have the estimate

$$H_\Theta^+ (g) \leq \sup_{\tau \geq 1} |H_{u_\tau}^+ g| + \sup_{\substack{\tau \geq 1 \\ j > j_{\tau-1}}} \left| \sum_{j \leq \ell < j_\tau} R_\ell (\tilde{R}_{\tau-1} g) \right| \leq H_{\Theta'}^+ (g) + \sup_{\tau \geq 1} H_{\Theta_\tau}^+ (\tilde{R}_{\tau-1} g)$$

as desired.  $\square$

**2.3. Maximal functions and cone frequency projections in  $D$ -lacunary directions.** We need to recall two interconnected results, both essentially due to Sjögren and Sjölin [24, Corollary 2.4]. Firstly, as mentioned in the introduction, for each nonnegative integer  $D$  and each  $1 < p \leq \infty$  the directional maximal function  $M_\Theta$  satisfies the bound

$$(2.1) \quad \|M_\Theta\|_{L^p \rightarrow L^p} \lesssim_{p,D} 1.$$

In particular the implicit constant is uniform over all  $D$ -lacunary sets of directions  $\Theta$ .

Consider now a  $D$ -lacunary set of directions  $\Theta = \{v_j\}_{j \geq 1}$  and the cone frequency projections  $R_j$ , as in the previous paragraph. Then, uniformly over all  $D$ -lacunary  $\Theta \subset S^1$  we have

$$(2.2) \quad \sup_{\varepsilon_j \in \{-1, 0, 1\}} \left\| \sum_j \varepsilon_j R_j f \right\|_{L^p(\mathbb{R}^2)} \leq \kappa_p(D) \|f\|_{L^p(\mathbb{R}^2)}.$$

We stress here that  $\kappa_p(D)$  only depends on  $p$  and the order of lacunarity  $D$ . The estimate above is contained within [24, Corollary 2.4]; see also [9, Lemma 3.3] for the case  $D = 1$ .

**2.4. Maximal Fourier multipliers and square functions.** The strategy for proving our main results is to first reduce the norm inequality for  $H_{\Theta}f$  to an appropriate square function estimate that will allow us to independently handle Littlewood-Paley pieces of the function  $f$ .

In order to formulate the main square function estimate we first introduce the standard Littlewood-Paley projections. To that end, let  $\phi$  be a Schwartz function supported on  $[1/2, 2]$  and such that

$$\sum_{k \in \mathbb{Z}} \phi(2^{-k}\xi) = 1, \quad \forall \xi \in \mathbb{R} \setminus \{0\}.$$

For a function  $f \in C_0^\infty(\mathbb{R}^2)$  set  $\widehat{S_k f}(\xi) := \widehat{f}(\xi)\phi(2^{-k}|\xi|)$ . The following proposition is a consequence of the Chang-Wilson-Wolff inequality [5]. The proof essentially follows from [12, Lemma 3.1]; see also [9, Proposition 3.1]. In the formulation that follows each operator  $P_j$  is a Fourier multiplier operator,  $\widehat{P_j f}(\xi) := m_j(\xi)\widehat{f}(\xi)$ .

**Proposition 2.2.** *Let  $\{P_j\}_{j=1}^N$  be Fourier multiplier operators which for all  $j$  satisfy  $|\partial_\xi^\alpha m_j(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}$  for all multi-indices  $\alpha$ , with  $C_\alpha$  independent of  $j$ . Then for all  $1 < p < \infty$  there exists  $C_p > 0$ , depending only on  $p$ , such that*

$$\left\| \sup_{j=1, \dots, N} |P_j f| \right\|_{L^p(\mathbb{R}^2)} \leq C_p (\log N)^{\frac{1}{2}} \left\| \left( \sum_{k \in \mathbb{Z}} \left( \sup_{j=1, \dots, N} |P_j(S_k f)| \right)^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)}.$$

**2.5. A single annulus pointwise estimate.** The next lemma relates the maximal operator associated with a 1-lacunary set, when localized to a frequency annulus, to the maximal function in the limiting direction. The proof is a simpler version of the argument used for the proof of [9, Lemma 3.2].

**Lemma 2.3.** *Assume that  $\{v_j\}_{j \geq 1}$  is a 1-lacunary set with limit  $v$ , following the conventions of §2.1. Let  $f \in \mathcal{S}(\mathbb{R}^2)$  with  $\text{supp } \widehat{f} \subseteq \{\xi \in \mathbb{R}^2 : \xi \cdot v_1 \geq 0, \xi \cdot v_\infty < 0\}$ . With  $H_v^+$  defined as in (1.1) we have for every  $k \in \mathbb{Z}$*

$$\sup_j |H_{v_j}^+(S_k f)(x)| \lesssim M_v[(S_k f)_{\text{odd}}](x) + M_v[(S_k f)_{\text{ev}}](x)$$

where  $g_{\text{odd}} := \sum_{j \text{ odd}} R_j g$  and similarly for  $g_{\text{ev}}$ .

**2.6. A Fefferman-Stein inequality for directional maximal functions.** Let  $\Theta = \{v_j\}_j$  be a  $D$ -lacunary set of directions for some nonnegative integer  $D$ . The following Fefferman-Stein type inequality will play an important role in deducing the square function estimate, which is the main estimate for the proof of Theorem 1. We include a proof for completeness although we do not claim any originality on this result.

**Lemma 2.4.** *Let  $\Theta = \{v_j\}_{j \geq 1}$  be a  $D$ -lacunary set of directions. For all  $1 < p, q < \infty$  we have*

$$\|\{M_{v_j} h_j\}\|_{L^p(\mathbb{R}^2; \ell^q)} \leq K_{q,p}(D) \|\{h_j\}\|_{L^p(\mathbb{R}^2; \ell^q)}$$

where the constant  $K_{q,p}(D)$  depends only on  $q, p, D$ .

*Proof.* It suffices to prove the same bound for the linearized operators

$$f \mapsto \frac{1}{2\varepsilon(x)} \int_{-\varepsilon(x)}^{\varepsilon(x)} |f(x + tu_j)| dt$$



which we continue to denote by  $M_{v_j}$ , uniformly over all positive measurable functions  $\varepsilon(x)$ . This formulation has the advantage of allowing for complex interpolation. The case  $p \leq q$  of the estimate of the lemma follows by complex interpolation of the following estimates for the vector valued operator  $\{h_j\} \mapsto \{M_{v_j}h_j\}$ ; namely the trivial estimate

$$\|\{M_{v_j}h_j\}\|_{L^p(\mathbb{R}^2; \ell^p)} \leq \|M_\Theta\|_{L^p \rightarrow L^p} \|\{h_j\}\|_{L^p(\mathbb{R}^2; \ell^p)} \lesssim_{p,D} \|\{h_j\}\|_{L^p(\mathbb{R}^2; \ell^p)},$$

with the easy estimate

$$\|\{M_{v_j}h_j\}\|_{L^p(\mathbb{R}^2; \ell^\infty)} \leq \|M_\Theta(\sup_j |h_j|)\|_{L^p(\mathbb{R}^2)} \lesssim_{p,D} \|\{h_j\}\|_{L^p(\mathbb{R}^2; \ell^\infty)}.$$

Observe that for  $\theta = 1 - p/q$  we have  $\theta \in [0, 1)$  and  $[L^p(\mathbb{R}^2; \ell^p) : L^p(\mathbb{R}^2; \ell^\infty)]_\theta = L^p(\mathbb{R}^2; \ell^q)$ .

We now prove the case  $p > q$ . Let  $r = (p/q)' > 1$ . We start by observing that

$$\|\{M_{v_j}h_j\}\|_{L^p(\mathbb{R}^2; \ell^q)}^q = \left( \int_{\mathbb{R}^2} \left( \sum_j |M_{v_j}h_j(x)|^q \right)^{\frac{p}{q}} dx \right)^{\frac{q}{p}} = \int_{\mathbb{R}^2} \sum_j |M_{v_j}h_j(x)|^q u(x) dx$$

for some nonnegative  $u \in L^r(\mathbb{R}^2)$  with  $\|u\|_r = 1$ . Given a Schwartz function  $f$  and a nonnegative function  $w$ , the Fefferman-Stein inequality

$$\int_{\mathbb{R}^2} |M_{v_j}f|^q w \lesssim_q \int_{\mathbb{R}^2} |f|^q M_{v_j}w$$

follows easily by its one-dimensional counterpart along the direction  $v_j \in S^1$ . Note that the estimates produced that way are independent of  $v_j \in S^1$ . Thus we can estimate

$$\begin{aligned} \int_{\mathbb{R}^2} \left( \sum_j |M_{v_j}h_j(x)|^q \right) u(x) dx &\lesssim_q \sum_j \int_{\mathbb{R}^2} |h_j(x)|^q M_{v_j}u(x) dx \leq \int_{\mathbb{R}^2} \sum_j |h_j(x)|^q M_\Theta u(x) dx \\ &\leq \left( \int_{\mathbb{R}^2} \left( \sum_j |h_j(x)|^q \right)^{\frac{p}{q}} dx \right)^{\frac{q}{p}} \|M_\Theta u\|_{L^r(\mathbb{R}^2)} \\ &\leq \|\{h_j\}\|_{L^p(\mathbb{R}^2; \ell^q)}^q \|M_\Theta\|_{L^r \rightarrow L^r} \|u\|_{L^r(\mathbb{R}^2)} \lesssim_{p,q,D} \|\{h_j\}\|_{L^p(\mathbb{R}^2; \ell^q)}^q \end{aligned}$$

which gives the desired Fefferman-Stein inequality for  $p > q$  by taking  $q$ -th roots on both sides of the estimate.  $\square$

### 3. PROOF OF THEOREM 1

This section is devoted to the proof of Theorem 1. Let us immediately observe that the upper bound of the theorem together with Karagulyan's  $L^2 \rightarrow L^2$  lower bound from [16] and Riesz-Thorin interpolation immediately implies the lower bound for all  $p \in (1, \infty)$ . Thus it suffices to prove the upper bound in Theorem 1 for all  $1 < p < \infty$ . We will actually prove the theorem for the operator  $H_\Theta^+$  as defined in (1.1) which is a linear combination of the identity and  $H_\Theta$ , and thus obeys equivalent norm bounds.

**3.1. Initial reductions.** Throughout, we fix a  $D$ -lacunary set of directions  $\Theta = \{v_j\}_{j \geq 1}$  with sufficiently large lacunarity constant which is a successor of a  $(D - 1)$ -lacunary set  $'\Theta = \{u_\tau\}_{\tau \geq 1}$ , so that  $\Theta = \cup_{\tau \geq 1} \Theta_\tau$ , according to the conventions and notations of §2.1. With these assumptions taken as understood we may reduce to the case where  $\widehat{f}$  is supported on the second and fourth quadrant. Indeed, since we assume that the directions of  $\Theta$  are in the first quadrant, the operators  $H_{v_i}^+$  do not touch functions with frequency support in the first and third quadrants. By symmetry, we can further reduce to the case that  $f$  has frequency support in the second quadrant.

Applying Proposition 2.2 to the family  $\{H_v^+\}_{v \in \Theta}$ , the upper bound in Theorem 1 reduces to the square function estimate

$$(3.1) \quad \left\| \left( \sum_{k \in \mathbb{Z}} |H_\Theta^+(S_k f)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \leq C_p(D) \|f\|_{L^p(\mathbb{R}^2)}, \quad 1 < p < \infty,$$

for any countable set of directions  $\Theta$  which is  $D$ -lacunary, where the constant  $C_p(D)$  depends only on  $p, D$ . Here we remember that  $S_k$  are smooth frequency projections to the annulus  $\{|\xi| \simeq 2^k\}$ .

**3.2. Proof of (3.1).** The main part of the proof proceeds by induction of (3.1) on the order of lacunarity. For  $D = 1$  the base step of the induction follows from [9, §3.2]. Now let  $D \geq 2$  and assume that (3.1) holds uniformly for all  $(D - 1)$ -lacunary sets of directions. The recurrence estimate of Lemma 2.1 applied to  $g = S_k f$  gives

$$H_\Theta^+ S_k f(x) \leq H_\Theta^+ S_k f + \sup_{\tau \geq 1} H_{\Theta_\tau}^+ (\tilde{R}_{\tau-1} S_k f).$$

This together with the inductive hypothesis now imply that

$$(3.2) \quad \left\| \left( \sum_{k \in \mathbb{Z}} |H_\Theta(S_k f)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \lesssim C_p(D-1) \|f\|_{L^p(\mathbb{R}^2)} + \left\| \left( \sum_{k \in \mathbb{Z}} \sup_{\tau \geq 1} |H_{\Theta_\tau}^+(\tilde{R}_{\tau-1} S_k f)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)}.$$

We now proceed with the estimate of the second summand in the right hand side above. According to Lemma 2.3, we have for each  $\tau \geq 1$  that

$$(3.3) \quad \begin{aligned} H_{\Theta_\tau}^+(\tilde{R}_{\tau-1} S_k f) &\lesssim M_{u_\tau}[(\tilde{R}_{\tau-1} S_k f)_{\text{odd}}] + M_{u_\tau}[(\tilde{R}_{\tau-1} S_k f)_{\text{ev}}] \\ &\lesssim \left( \sum_{\tau \geq 1} |M_{u_\tau}(F_1^{k,\tau})|^2 \right)^{\frac{1}{2}} + \left( \sum_{\tau \geq 1} |M_{u_\tau}(F_2^{k,\tau})|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where  $F_1^{k,\tau} := (\tilde{R}_{\tau-1} S_k f)_{\text{odd}}$  and  $F_2^{k,\tau} := (\tilde{R}_{\tau-1} S_k f)_{\text{even}}$ . We only treat the first summand in the estimate above, the estimate for the second one being similar. Observe that

$$F_1^{k,\tau} = \sum_{j \text{ odd}} R_j(\tilde{R}_{\tau-1} S_k f) = \sum_{j \in J_\tau} R_j(S_k f)$$

where  $J_\tau = \{j \in \Theta_\tau : j \text{ odd}\}$ . Applying the case  $q = 2$  of the Fefferman-Stein inequality of Lemma 2.4 to the  $(D - 1)$ -lacunary sequence  $\{u_\tau\}_{\tau \geq 1}$  yields

$$\left\| \left( \sum_{k \in \mathbb{Z}} \sum_{\tau \geq 1} |M_{u_\tau}(F_1^{k,\tau})|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \leq K_p(D - 1) \left\| \left( \sum_k \sum_\tau |F_1^{k,\tau}|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)}.$$

However, the right hand side of the previous estimate can be estimated by the biparameter Khintchine inequality as follows

$$\begin{aligned} \left\| \left( \sum_k \sum_\tau |F_1^{k,\tau}|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)}^p &\lesssim \mathbb{E}_\eta \mathbb{E}_\omega \int_{\mathbb{R}^2} \left| \sum_{\tau \geq 1} \sum_{j \in J_\tau} \varepsilon_\tau(\eta) R_j \left( \sum_{k \in \mathbb{Z}} \varepsilon_k(\omega) S_k f \right) \right|^p \\ &\leq \kappa_p(D - 1)^p \mathbb{E}_\omega \int_{\mathbb{R}^2} \left| \sum_{k \in \mathbb{Z}} \varepsilon_k(\omega) S_k f \right|^p \simeq_p \kappa_p(D - 1)^p \|f\|_{L^p(\mathbb{R}^2)}^p, \end{aligned}$$

where we used (2.2) in going from the first to the second line and the Littlewood-Paley inequality in the last estimate. Arguing similarly for  $F_2^{k,\tau}$  we have that

$$\sum_{\text{par} \in \{1,2\}} \left\| \left( \sum_{k \in \mathbb{Z}} \sum_{\tau \geq 1} |M_{u_\tau}(F_{\text{par}}^{k,\tau})|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \lesssim_{D,p} \|f\|_{L^p(\mathbb{R}^2)}$$

Using these estimates in (3.3) and replacing in (3.2) yields (3.1) and thus completes the proof of Theorem 1.

#### 4. PROOF OF THEOREM 2

This section is devoted to the proof of Theorem 2. We begin with some reductions of the problem which follow closely the arguments in [15, §3].

**4.1. Preliminary reductions.** We begin by setting, for  $d = 1, \dots, D$ ,

$$\mathbf{v}_d(x) := \prod_{j=1}^d e^{2\pi i 2^{\lfloor \log_2 \lambda_j(x) \rfloor}}, \quad \Theta_d := \mathbf{v}_d(\mathbb{R}^2),$$

so  $\Theta_d$  is the range of  $\mathbf{v}_d$  for each  $d$ . Notice that, by splitting  $f$  into a finite number of disjointly supported pieces, and rotating each piece by a fixed angle, we can assume that  $\lambda_1 : \mathbb{R}^2 \rightarrow (0, 2^{-15}]$ . Then  $\lambda_j \leq 2^{-5(j-1)} 2^{-15}$  for each  $j \geq 2$  and each  $\Theta_d$ ,  $d = 1, \dots, D$ , is contained in the first quadrant portion of the cone  $C_{\text{nh}}$  bordered by the directions  $\{e^{2\pi i 0}, e^{2\pi i 2^{-5}}\}$ . We later refer to sets of directions  $\Theta$  contained in  $C_{\text{nh}}$  as being *nearly horizontal*.

Consider the cone in the second quadrant

$$\Gamma_0 := \{\xi \in \mathbb{R}^2 : \xi_2 > -2\xi_1, \xi_1 < -2^6, \xi_2 > 0\}.$$

The arguments of [15, §3], which apply to any vector field with nearly horizontal range, yield the pointwise bound

$$|H_{\mathbf{v}_D, 1} f| \lesssim M_{(0,1)} M_{(1,0)} f + M_{\Theta_D} f$$

whenever  $\widehat{f}$  is supported off  $\Gamma_0 \cup -\Gamma_0$ . As the right hand side of the last display is  $L^p$ -bounded, and by treating similarly the part supported in the lower half-plane, we are allowed to assume that  $\text{supp } \widehat{f} \subset \Gamma_0$  for the remainder of the proof. In this case, following again [15], we have

$$\|H_{\mathbf{v}_D,1}f\|_p \lesssim \sup_{|\tau| \leq 2^4} \|H_{\mathbf{v}_D}^{+\tau}f\|_p + \|M_{\Theta_D}f\|_p$$

where we are using the notation, for a generic direction  $v$  and vector field  $\mathbf{v}$ ,

$$H_v^{+\tau}f(x) = \int_{\mathbb{R}^2} \widehat{f}(\xi) \mathbf{1}_{[\tau,\infty)}(\xi \cdot v) e^{2\pi i x \cdot \xi} d\xi, \quad H_{\mathbf{v}(x)}^{+\tau}f(x) := H_{\mathbf{v}(x)}^{+\tau}f(x).$$

We have thus reduced Theorem 2 to the proof of the estimate

$$(4.1) \quad \|H_{\mathbf{v}_D}^{+\tau}f\|_p \leq \beta_p(D) \|f\|_p$$

where the constant  $\beta_p(D)$  can be chosen uniform over  $|\tau| \leq 2^4$ , whenever  $\text{supp } \widehat{f} \subset \Gamma_0$ . In order to simplify the notation we will work in the paradigmatic case  $\tau = 0$ , so we only show (4.1) for  $H^+$ . However, the proof for  $H^{+\tau}$  is essentially identical and the details are left to the reader.

**Remark 4.1.** In what follows we will consider the Hilbert transform  $H_v f$  along almost horizontal directions  $v$ . Now it is not hard to verify that  $H_v f(x)$  does not depend on  $v$  when  $f$  has frequency support in  $\{\xi_2 \leq -2\xi_1\}$ . Thus, instead of the cone  $\Gamma_0$  we could have considered the cone  $\Gamma_1 := \{\xi_1 < -2^6, \xi_2 > 0\}$ .

A consequence of the definition of  $\Gamma_0$  is that for every  $\xi \in \Gamma_0$  and every almost horizontal direction  $v$  we have that

$$(4.2) \quad |\xi| \leq 2\xi \cdot v^\perp.$$

**4.2. Reduction of (4.1) to a lacunary vector-valued estimate.** In the next paragraph, we will prove the following proposition.

**Proposition 4.2.** *Let  $\Theta = \{v_\theta\}_{\theta \in \Theta} = \{e^{2\pi i \theta}\}_{\theta \in \Theta}$  be a finite almost horizontal  $D$ -lacunary set of directions. For each  $\theta \in \Theta$ , let  $\lambda_\theta : \mathbb{R}^2 \rightarrow (0, 2^{-15}]$  be Lipschitz functions with  $\|u_\theta\|_{\text{LIP}} \leq 1$  and define*

$$\mathbf{v}_\theta : \mathbb{R}^2 \rightarrow S^1, \quad \mathbf{v}_\theta(x) := e^{2\pi i \theta} e^{2\pi i 2^{\lfloor \log_2 \lambda_\theta(x) \rfloor}}.$$

For every  $p \in (1, \infty)$  there exists a constant  $B_p(D)$ , depending only on  $p$  and  $D$ , such that

$$\| \{H_{\mathbf{v}_\theta} f_\theta\} \|_{L^p(\mathbb{R}^2; \ell^2(\Theta))} \leq B_p(D) \| \{f_\theta\} \|_{L^p(\mathbb{R}^2; \ell^2(\Theta))}, \quad 1 < p < \infty,$$

whenever the vector  $\{f_\theta\}_{\theta \in \Theta}$  has frequency support in  $\Gamma_0$ .

Before the proof of Proposition 4.2, let us see how the proposition is used to establish (4.1). First observe that the assumption  $\lambda_{j-1}(x) \leq 2^{-5} \lambda_j(x)$  and implies that  $\Theta_D = \mathbf{v}_D(\mathbb{R}^2)$  is a  $D$ -lacunary set which is a successor of  $\Theta_{D-1} = \mathbf{v}_{D-1}(\mathbb{R}^2)$ . Let us write  $\Theta_D = \{e^{2\pi i \theta_j}\}_{j \geq 1}$  and  $\Theta_{D-1} = \{e^{2\pi i \gamma_\tau}\}_{\tau \geq 1}$  so that  $\Theta_D$  is a successor to  $\Theta_{D-1} = \Theta_D$ . Using the notations of §2.1 we can then write  $\Theta_D = \cup_{\tau \geq 1} \Theta_{D,\tau}$  where each  $\Theta_{D,\tau}$  is a 1-lacunary set with limit  $e^{2\pi i \gamma_\tau}$ . It is essential to note at this point that, if  $v_j := e^{2\pi i \theta_j} \in \Theta_{D,\tau}$  for some (unique)  $\tau$  and  $e^{2\pi i \theta_j} = \mathbf{v}_D(x)$  then  $u_\tau := e^{2\pi i \gamma_\tau} = \mathbf{v}_{D-1}(x)$  and

$$e^{2\pi i \theta_j} = e^{2\pi i \gamma_\tau} e^{2\pi i 2^{\lfloor \log_2 \lambda_D(x) \rfloor}} =: e^{2\pi i \gamma_\tau} \alpha(x).$$

Now let  $v_j \in \Theta_D$  so that  $v_j \in \Theta_{D,\tau}$  for a unique  $\tau \geq 1$  and let  $x$  be such that  $\mathbf{v}_D(x) = v_j$  and  $\mathbf{v}_{D-1}(x) = u_\tau$ . By the previous considerations and an inspection of the proof of Lemma 2.1 we get for such  $x$  fixed

$$\begin{aligned} |H_{\mathbf{v}_D(x)}^+ f(x)| &= |H_{v_j}^+ f(x)| \leq |H_{u_\tau}^+ f(x)| + |H_{v_j}^+ \tilde{R}_{\tau-1} f(x)| \\ &= |H_{\mathbf{v}_{D-1}(x)}^+ f(x)| + |H_{u_\tau \alpha(x)}^+ \tilde{R}_{\tau-1} f(x)| \\ &\leq |H_{\mathbf{v}_{D-1}(x)}^+ f(x)| + \left( \sum_{\tau \geq 1} |H_{u_\tau \alpha(x)}^+ \tilde{R}_{\tau-1} f(x)|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

We now use the above relation to find inductively  $\beta_p(D)$  so that (4.1) holds. If  $D = 0$ , then  $H_{\mathbf{v}_D}^+ = H_{e^{2\pi i 0}}^+$  is the usual Hilbert transform on  $\mathbb{R}$ , therefore (4.1) holds with  $\beta_p(0) \simeq \max(p, p/(p-1))$ . Suppose now that  $\beta_p(D-1) < \infty$ . Using the previous estimate together with Proposition 4.2 applied to the  $(D-1)$ -lacunary set  $\Theta_{D-1} = \{e^{2\pi i y_\tau}\}_{\tau \geq 1}$  and the vector field  $\mathbf{v}_\tau(x) := e^{2\pi i y_\tau} \alpha(x)$ , we get

$$\begin{aligned} (4.3) \quad \|H_{\mathbf{v}_D}^+ f\|_p &\leq \|H_{\mathbf{v}_{D-1}} f\|_{L^p(\mathbb{R}^2)} + \|\{H_{u_\tau \alpha(x)}(\tilde{R}_{\tau-1} f)\}\|_{L^p(\mathbb{R}^2; \ell_\tau^2)} \\ &\leq \beta_p(D-1) \|f\|_{L^p(\mathbb{R}^2)} + B_p(D-1) \|\{\tilde{R}_{\tau-1} f\}\|_{L^p(\mathbb{R}^2; \ell_\tau^2)} \\ &\leq [\beta_p(D-1) + B_p(D-1) \kappa_p(D-1)] \|f\|_{L^p(\mathbb{R}^2)}, \end{aligned}$$

where we applied the Sjögren-Sjolin cone multiplier estimate (2.2) in the last inequality. So we may take  $\beta_p(D)$  to be the bracketed expression in the last display. This completes the induction in the proof of (4.1) and therefore the proof of Theorem 2, up to showing the vector valued estimate of Proposition 4.2.

**4.3. Proof of Proposition 4.2.** All the implicit constants appearing below are allowed to depend on the lacunarity order  $D$  and on exponents  $p$  without explicit mention.

**4.3.1. Littlewood-Paley projections and preliminary reductions.** Throughout the proof, we will adopt the following notation. Let  $\phi, \psi$  be even real Schwartz functions on  $\mathbb{R}$  satisfying the following assumptions:  $\widehat{\phi}$  is supported on  $\{1 \leq |t| \leq 2\}$  and normalized so that

$$\sum_k \widehat{\phi}(2^{-k}t) = 1, \quad t \in \mathbb{R} \setminus \{0\},$$

while  $\psi$  is spatially supported on  $[-1, 1]$  and normalized so that

$$\int_{\mathbb{R}} \psi = 0, \quad \sum_k |\widehat{\psi}(2^{-k}t)|^2 = 1, \quad t \in \mathbb{R} \setminus \{0\}.$$

Denote by  $\phi_k(\cdot) = 2^k \phi(2^k \cdot)$ ,  $\psi_k(\cdot) := 2^k \psi(2^k \cdot)$ . For a direction  $\theta \in S^1$ , we then introduce the corresponding Littlewood-Paley projections in the direction  $\theta$

$$\begin{aligned} \Phi_{\theta,k} f(x) &:= \int_{\mathbb{R}} f(x - te^{2\pi i \theta}) \phi_k(t) dt, \\ \Psi_{\theta,k} f(x) &:= \int_{\mathbb{R}} f(x - te^{2\pi i \theta}) \psi_k(t) dt, \quad \widetilde{\Psi}_{\theta,k} := \Psi_{\theta,k} \circ \Psi_{\theta,k}. \end{aligned}$$

We will also use the cutoffs

$$A_{\theta,k}f(x) := \sum_{\tau \leq k} \int_{\mathbb{R}} f(x - te^{2\pi i\theta}) \phi_{\tau}(t) dt, \quad B_{\theta,k} := \text{Id} - A_{\theta,k}.$$

Note that the frequency support of  $A_{\theta,k}$  is contained in a band of width  $2^{k+2}$  around the line through the origin oriented along  $\theta$ , and which acts as the identity on functions with frequency support contained in a band of width  $2^k$  and same orientation. Finally we denote for  $j \in \mathbb{N}$  and  $x \in \mathbb{R}^2$

$$v_{\theta,j} := e^{2\pi i\theta} e^{2\pi i2^{-j}}, \quad j_{\theta}(x) := -\lfloor \log_2 \lambda_{\theta}(x) \rfloor, \quad E_{\theta,j} := \{x \in \mathbb{R}^2 : j_{\theta}(x) = j\}.$$

The assumptions on  $\phi, \psi$  together with the hypothesis  $\text{supp } \widehat{f_{\theta}} \subset \Gamma_0$  guarantee that for all  $\theta, j$

$$f_{\theta} = \sum_{k \geq 5} \sum_{\ell \in \mathbb{Z}} \Phi_{\theta,k} \widetilde{\Psi}_{\theta^{\perp}, k+j+\ell} f_{\theta}.$$

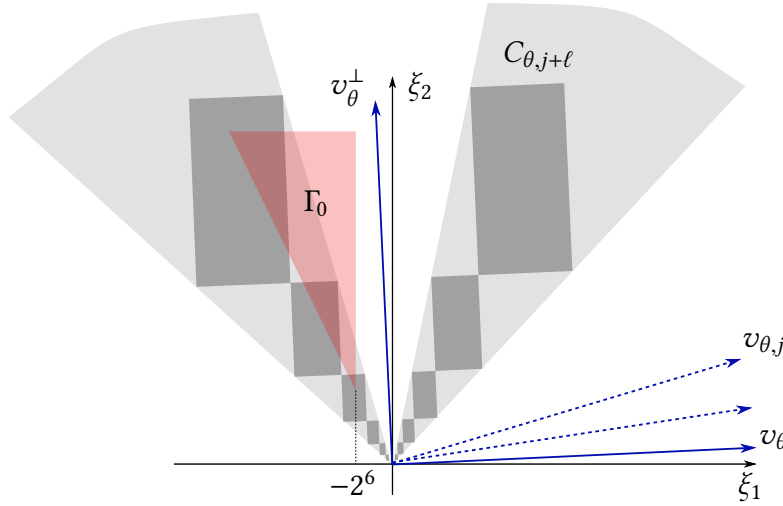


FIGURE 2. A figure for the proof of Proposition 4.2. The dark shaded rectangles represent the approximate frequency supports of  $\Delta_{\theta,j,\ell,k}$ .

4.3.2. *The initial decomposition.* With the notation above we immediately get the identity

$$H_{v_{\theta}} f_{\theta} = \sum_{j \geq 1} \mathbf{1}_{E_{\theta,j}} H_{v_{\theta,j}} f_{\theta};$$

introducing without loss of generality the qualitative assumption that  $\inf_{\theta} \inf_{x \in \mathbb{R}^2} \lambda_{\theta} > 0$ , we obtain that the sum in  $j$  above is over a finite range  $J \subset \mathbb{N}$ . Proceeding as in [15, §3], we obtain the decomposition

$$(4.4) \quad H_{v_{\theta}(x)} f(x) = \sum_{\ell \in \mathbb{Z}} H_{v_{\theta,j}} \Delta_{\theta,j,\ell} f_{\theta}(x) + \sum_{k \geq 5} \sum_{\tau \geq k+j+5} \Phi_{\theta,k} \widetilde{\Psi}_{\theta^{\perp},\tau} f_{\theta}(x), \quad x \in E_{\theta,j},$$

with

$$(4.5) \quad \Delta_{\theta,j,\ell} := \sum_{k \geq 5} \Delta_{\theta,j,\ell,k}, \quad \Delta_{\theta,j,\ell,k} := \begin{cases} B_{\theta^\perp, k+j+\lfloor \ell/2 \rfloor} \Phi_{\theta,k} \widetilde{\Psi}_{\theta^\perp, k+j+\ell}, & \ell \leq -5, \\ \Phi_{\theta,k} \widetilde{\Psi}_{\theta^\perp, k+j+\ell}, & -4 \leq \ell \leq 4, \\ A_{\theta^\perp, k+j+\lfloor \ell/2 \rfloor} \Phi_{\theta,k} \widetilde{\Psi}_{\theta^\perp, k+j+\ell}, & \ell \geq 5. \end{cases}$$

Replacing the disjoint support in  $j$  of  $\mathbf{1}_{E_{\theta,j}}$  by a square function in  $j$  in the first summand of the right hand side of (4.4) we get

$$(4.6) \quad \begin{aligned} & \| \{H_{\nu_\theta} f_\theta\} \|_{L^p(\mathbb{R}^2; \ell^2(\Theta))} \\ & \leq \sum_{\ell \in \mathbb{Z}} \| \{H_{\nu_{\theta,j}} \Delta_{\theta,j,\ell} f_\theta\} \|_{L^p(\mathbb{R}^2; \ell^2(\Theta \times J))} + \left\| \left\{ \sum_{k \geq 5} \sum_{\tau \geq k+j_\theta(\cdot)+5} \Phi_{\theta,k} \widetilde{\Psi}_{\theta^\perp, \tau} f_\theta \right\} \right\|_{L^p(\mathbb{R}^2; \ell^2(\Theta))}. \end{aligned}$$

**Remark 4.3.** For the purposes of this section we may precompose all multipliers with a smooth Fourier restriction adopted to the cone  $\Gamma_0$ . Therefore, making use of the smoothness and vanishing of the Fourier transform of  $\psi$ , and relying upon relation (4.2), we obtain the symbol estimates

$$\sup_{\theta \in \Theta} \left( \sup_{|\alpha| \leq 100} |2^{-k} \xi|^{|\alpha|} |\partial_\xi^\alpha \widehat{\Psi}_{\theta^\perp, k}(\xi)| + |2^{-k} \xi|^{-1} |\widehat{\Psi}_{\theta^\perp, k}(\xi)| \right) \lesssim 1, \quad k \in \mathbb{Z}.$$

In particular, because of the uniformity of the Hörmander-Mikhlin condition above in  $\theta$ , the vector-valued multiplier operators

$$\ell^q \ni \{f_\theta\}_\theta \mapsto \left\{ \sum_k \varepsilon_k \Psi_{\theta^\perp, k} f_\theta \right\}_\theta$$

are bounded on  $L^p(\mathbb{R}^2; \ell^q)$ ,  $1 < p, q < \infty$ , uniformly over choices of  $\varepsilon_k \in \{-1, 0, 1\}$ . A similar reasoning applies to  $\widetilde{\Psi}_{\theta^\perp, k}$  and (simpler) to  $\Phi_{\theta^\perp, k}$ . In fact, in view of (4.2) we could have worked with a radial frequency projection (with or without compact support) independent of  $\theta$  in place of the directional projections along  $\theta^\perp$ . However, certain geometric considerations made in [15] and of use to us are more naturally expressed in terms of directional frequency projections of the type  $\Psi_{\theta^\perp, k}$ .

4.3.3. *Estimation of the first summand in (4.6).* In order to estimate the first summand, which is an error term, we will show that each term in the sum has sufficient decay with respect to the parameter  $\ell$ . We will need two tools, both of which are consequences of the  $L^p$ -boundedness of the directional maximal function  $M_{\Theta'}$  along  $\Theta' := \Theta \cup \bigcup_{\theta} \{v_{\theta,j}\}$ , which holds since  $\Theta'$  is a  $(D+1)$ -lacunary set; see (2.1). The first is the Córdoba-Fefferman inequality, [7], in the form given in [10, Theorem 6.1],

$$(4.7) \quad \| \{H_{\nu_{\theta,j}} g_{\theta,j}\} \|_{L^p(\mathbb{R}^2; \ell^2(\Theta \times J))} \lesssim \| \{g_{\theta,j}\} \|_{L^p(\mathbb{R}^2; \ell^2(\Theta \times J))},$$

which we will apply to the doubly indexed sequence  $g_{\theta,j} = \Delta_{\theta,j,\ell} f_\theta$ . Further, we claim the inequality

$$(4.8) \quad \| \{ \Delta_{\theta,j,\ell} f_\theta \} \|_{L^p(\mathbb{R}^2; \ell^2(\Theta \times J))} \lesssim \omega_\ell \| \{f_\theta\} \|_{L^p(\mathbb{R}^2; \ell^2(\Theta))},$$

where  $\omega_\ell$  is a suitable rapidly decaying sequence. We detail the proof in §4.3.7 below but sketch the heuristic here. The operators  $\Delta_{\theta,\ell,j}$  are approximate restrictions to the frequency cones

$$(4.9) \quad C_{\theta,\mu} := \{ \xi \in \mathbb{R}^2 : 2^{-\mu-1} |\xi \cdot e^{2\pi i \theta^\perp}| < |\xi \cdot e^{2\pi i \theta}| < 2^{-\mu} |\xi \cdot e^{2\pi i \theta^\perp}| \}$$

for  $\mu = \ell + j$ , so that, up to Schwartz tails  $\Delta_{\theta,j,\ell} f_\theta = \Delta_{\theta,j,\ell} f_{\theta,j,\ell}$  where  $f_{\theta,j,\ell}$  denotes the rough frequency restriction of  $f_\theta$  to  $C_{\theta,j+\ell}$ ; see Figure 2. We will see that the  $\Delta_{\theta,j,\ell}$  (and suitable Schwartz-tail free modifications) satisfy the vector valued bound

$$(4.10) \quad \left\| \{ \Delta_{\theta,j,\ell} g_{\theta,j} \} \right\|_{L^p(\mathbb{R}^2; \ell^2(\Theta \times J))} \lesssim \omega_\ell \left\| \{ g_{\theta,j} \} \right\|_{L^p(\mathbb{R}^2; \ell^2(\Theta \times J))}$$

with the extra decay  $\omega_\ell$ , when  $\mp \ell \geq 5$ , due respectively to the smallness of the symbol of  $\tilde{\Psi}_{\theta^\perp, k+j+\ell}$  on the support of  $B_{\theta^\perp, k+j+\lfloor \ell/2 \rfloor}$ , and to the mean zero of  $\tilde{\Psi}_{\theta^\perp, k+j+\ell}$  coupled with  $A_{\theta^\perp, k+j+\lfloor \ell/2 \rfloor}$  being constant near zero. Finally, (4.8) will follow from the vector valued bound for the multipliers associated to the cones  $C_{\theta,j+\ell}$ . Combining (4.8) with (4.7) yields the required summable decay for the first term in (4.6).

4.3.4. *Estimation of the second summand in (4.6).* We linearize the  $L^p(\mathbb{R}^2; \ell^2)$ -norm of the second summand by multiplying and integrating against some  $\{g_\theta\}_{\theta \in \Theta}$  with  $\|\{g_\theta\}\|_{L^{p'}(\mathbb{R}^2; \ell^2(\Theta))} = 1$ . We expand one of the convolutions in  $\tilde{\Psi}_{\theta^\perp, \tau}$  and rewrite the second summand as

$$\sum_{\theta \in \Theta} \int_{\mathbb{R}^2} \sum_{k \geq 5} \int_{\mathbb{R}} \sum_{\tau \geq k+j_\theta(x)+5} \Phi_{\theta,k} \Psi_{\theta^\perp, \tau} f_\theta(x - te^{2\pi i \theta^\perp}) \psi_\tau(t) g_\theta(x) dt dx.$$

Now we split the expression in the last display into two parts; the main term

$$(4.11) \quad \sum_{\theta \in \Theta} \int_{\mathbb{R}^2} \sum_{k \geq 5} \int_{\mathbb{R}} \sum_{\tau \geq k+j_\theta(x-te^{2\pi i \theta^\perp})+5} \Phi_{\theta,k} \Psi_{\theta^\perp, \tau} f_\theta(x - te^{2\pi i \theta^\perp}) \psi_\tau(t) g_\theta(x) dt dx,$$

and the error term which up to a sign is equal to

$$(4.12) \quad \sum_{\theta \in \Theta} \int_{\mathbb{R}^2} \sum_{k \geq 5} \int_{\mathbb{R}} \sum_{j_\theta^-(x,t) < \tau - k - 5 \leq j_\theta^+(x,t)} \Phi_{\theta,k} \Psi_{\theta^\perp, \tau} f_\theta(x - te^{2\pi i \theta^\perp}) \psi_\tau(t) g_\theta(x) dt dx,$$

where  $j_\theta^\pm(x, t) := \pm \max\{\pm j_\theta(x), \pm j_\theta(x - te^{2\pi i \theta^\perp})\}$ .

4.3.5. *The main term coming from (4.11).* In the main term we may change variables in the  $dxdt$  integral to pass the Littlewood-Paley operator  $\Psi_{\theta, \tau}$  to  $g_\theta$  and later exchange summation order in  $\tau, k$  to obtain

$$\sum_{\theta \in \Theta} \int_{\mathbb{R}^2} \sum_{\tau \geq 10+j_\theta(x)} \sum_{5 \leq k \leq \tau - j_\theta(x) - 5} \Phi_{\theta,k} \Psi_{\theta^\perp, \tau} f_\theta(x) \Psi_{\theta^\perp, \tau} g_\theta(x) dx.$$

By  $\ell^2(\Theta)$ -valued Littlewood-Paley theory, see Remark 4.3,

$$\|\{\Psi_{\theta^\perp, \tau} g_\theta\}\|_{L^{p'}(\mathbb{R}^2; \ell^2(\mathbb{Z} \times \Theta))} \lesssim \|\{g_\theta\}\|_{L^{p'}(\mathbb{R}^2; \ell^2(\Theta))} = 1$$

therefore it suffices to estimate

$$(4.13) \quad \left\| \left\{ \sup_{r \geq 0} \left| \sum_{5 \leq k \leq 5+r} \Phi_{\theta,k} \Psi_{\theta^\perp, \tau} f_\theta \right| \right\} \right\|_{L^p(\mathbb{R}^2; \ell^2(\mathbb{Z} \times \Theta))} \lesssim \left\| \{ M_{e^{2\pi i \theta}}(\Psi_{\theta^\perp, \tau} f_\theta) \} \right\|_{L^p(\mathbb{R}^2; \ell^2(\mathbb{Z} \times \Theta))} \\ \lesssim \|\{\Psi_{\theta^\perp, \tau} f_\theta\}\|_{L^p(\mathbb{R}^2; \ell^2(\mathbb{Z} \times \Theta))} \lesssim \|\{f_\theta\}\|_{L^p(\mathbb{R}^2; \ell^2(\Theta))}.$$

The second estimate above follows by the Fefferman-Stein inequality of Lemma 2.4 and the last estimate follows again by the vector-valued  $L^p$ -bounds of Remark 4.3. This concludes the treatment of the main term.



4.3.6. *The error term coming from (4.12).* As noticed in [15], the support conditions on  $\psi_{\tau,\theta}$  and the Lipschitz condition on  $\lambda_\theta$  yields that

$$|\lambda_\theta(x - te^{2\pi i\theta}) - \lambda_\theta(x)| \leq |t| \leq 2^{-\tau} \leq 2^{-j_\theta^-(x,t)-10}.$$

It is then easy to see that  $j_\theta^+(x,t) \leq j_\theta^-(x,t) + 1$ . Hence, by changing variables in the same fashion as for the main term, the expression in (4.12) can be written in the form

$$\begin{aligned} & \sum_{\theta \in \Theta} \int_{\mathbb{R}^2} \sum_{10+j_\theta^- < \tau \leq 10+j_\theta^+} \left( \sum_{5 \leq k \leq \tau - j_\theta^- - 5} \Phi_{\theta,k} \Psi_{\theta^\perp, \tau} f_\theta(x) \right) \Psi_{\theta^\perp, \tau} g_\theta(x) \, dx \\ & + \sum_{\theta \in \Theta} \int_{\mathbb{R}^2} \sum_{10+j_\theta^+ < \tau} \left( \sum_{\tau - j_\theta^+ - 5 \leq k \leq \tau - j_\theta^- - 5} \Phi_{\theta,k} \Psi_{\theta^\perp, \tau} f_\theta(x) \right) \Psi_{\theta^\perp, \tau} g_\theta(x) \, dx, \end{aligned}$$

where we suppressed the dependence of  $j_\theta^\pm$  on  $x, t$ . These two terms are estimated by an argument similar to the one used for the main term. In particular, the estimate for the first summand reduces to the estimate

$$\left\| \left\{ \sup_{0 \leq r \leq 1} \left| \sum_{5 \leq k \leq 5+r} \Phi_{\theta,k} \Psi_{\theta^\perp, \tau} f_\theta \right| \right\} \right\|_{L^p(\mathbb{R}^2; \ell^2(\mathbb{Z} \times \Theta))} \lesssim \| \{ f_\theta \} \|_{L^p(\mathbb{R}^2; \ell^2(\Theta))}$$

which is proven in the same fashion as (4.13). The second summand reduces in turn to the estimate

$$\left\| \left\{ \sup_{\substack{r > 5 \\ 0 \leq s \leq 1}} \left| \sum_{r \leq k \leq r+s} \Phi_{\theta,k} \Psi_{\theta^\perp, \tau} f_\theta \right| \right\} \right\|_{L^p(\mathbb{R}^2; \ell^2(\mathbb{Z} \times \Theta))} \lesssim \| \{ f_\theta \} \|_{L^p(\mathbb{R}^2; \ell^2(\Theta))}$$

which is again estimated as in (4.13). This completes the treatment of the error term and the proof of Theorem 4.2.

4.3.7. *Proof of (4.8).* We begin with a further decomposition of the multipliers  $\Delta_{\theta,j,\ell}$  of (4.5). We write

$$\Delta_{\theta,j,\ell} = \sum_{\beta \geq 0} \Delta_{\theta,j,\ell}^\beta$$

where

$$\Delta_{\theta,j,\ell}^0 := \sum_{k \geq 5} A_{\theta^\perp, k+\ell+j} \Delta_{\theta,j,\ell,k}, \quad \Delta_{\theta,j,\ell}^\beta := \sum_{k \geq 5} \Phi_{\theta^\perp, k+\ell+j+\beta} \Delta_{\theta,j,\ell,k}, \quad \beta \geq 1.$$

If  $C_{\theta,\mu}$  is as in (4.9) and  $R_{\theta,\mu}$  is the corresponding Fourier restriction, it is now easy to verify that when  $\beta \geq 1$

$$\Delta_{\theta,j,\ell}^\beta = \Delta_{\theta,j,\ell}^\beta R_{\theta,\ell+j+\beta}.$$

The first step is the observation that for all  $\varepsilon > 0$  and weights  $u$  on  $\mathbb{R}^2$

$$(4.14) \quad \| \Delta_{\theta,j,\ell}^\beta g \|_{L^2(u)} \lesssim C_\varepsilon 2^{-\beta} \omega_\ell \| g \|_{L^2(w)}, \quad w := (M_{e^{2\pi i\theta}}^{\text{st}}(u^{1+\varepsilon}))^{\frac{1}{1+\varepsilon}},$$

where  $M_v^{\text{st}}$  stands for the strong maximal function in the coordinates of  $v$ . This stems from the observation that the dilation of the symbol  $\Delta_{\theta,j,\ell}^\beta$  in direction  $e^{2\pi i\theta}$  by a factor of  $2^{j+\ell+\beta}$  is a standard Hörmander-Mikhlin multiplier with symbol estimates controlled by  $\omega_\ell$  and with additional decay in  $\beta$  introduced from the Schwartz tails of  $\tilde{\Psi}_{\theta^\perp, k+\ell+j}$  on the support of  $\Phi_{\theta^\perp, k+\ell+j+\beta}$ . Therefore such symbol satisfies the well known analogue of (4.14) with the Hardy-Littlewood

maximal function in place of  $M_{e^{2\pi i\theta}}^{\text{st}}$ , namely the Córdoba-Fefferman inequality [6]; see also [22]. The estimate (4.14) readily yields the vector valued analogue

$$\|\{\Delta_{\theta,j,\ell}^{\beta} g_{\theta,j}\}\|_{L^2(u; \ell^2(\Theta \times J))} \lesssim C_{\varepsilon} 2^{-\beta} \omega_{\ell} \|\{g_{\theta,j}\}\|_{L^2(w; \ell^2(\Theta \times J))}, \quad w := (M_{\Theta}^{\text{st}}(u^{1+\varepsilon}))^{\frac{1}{1+\varepsilon}}.$$

The  $L^q$ -boundedness of  $M_{\Theta}^{\text{st}}$  for all  $1 < q < \infty$  and standard duality reasoning yield the inequality

$$\|\{\Delta_{\theta,j,\ell}^{\beta} g_{\theta,j}\}\|_{L^p(\mathbb{R}^2; \ell^2(\Theta \times J))} \lesssim 2^{-\beta} \omega_{\ell} \|\{g_{\theta,j}\}\|_{L^p(\mathbb{R}^2; \ell^2(\Theta \times J))}, \quad 1 < p < \infty$$

which we apply to  $g_{\theta,j} = R_{\theta,\ell+j+\beta} f_{\theta}$ , thus obtaining

$$\begin{aligned} \|\{\Delta_{\theta,j,\ell}^{\beta} f_{\theta,j}\}\|_{L^p(\ell^2(\Theta \times J))} &= \|\{\Delta_{\theta,j,\ell}^{\beta} R_{\theta,\ell+j+\beta} f_{\theta}\}\|_{L^p(\ell^2(\Theta \times J))} \\ &\lesssim 2^{-\beta} \omega_{\ell} \|\{R_{\theta,\ell+j+\beta} f_{\theta}\}\|_{L^p(\ell^2(\Theta \times J))} \lesssim 2^{-\beta} \omega_{\ell} \|\{f_{\theta}\}\|_{L^p(\ell^2(\Theta))}. \end{aligned}$$

The last step is  $\ell^2(\Theta)$ -valued Littlewood-Paley theory of the (lacunary) cone multiplier operators  $\{R_{\theta,\ell+j+\beta} : j \in J\}$ . Finally, a summation over  $\beta$  of the above inequality yields the claimed (4.8) and completes the proof.

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