# Can nontradables generate substantial home bias? Technical Appendix

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November 1999

# 1 A model of international portfolio choice with nontradables: extensions

In this Appendix we focus on the technical aspects of the model presented in the main text.

Domestic residents receive an endowment stream of nontradables $S(t)$, whose stochastic rate of growth is given by

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma_X d\xi(t) \tag{A.1}$$

where $\mu$ and $\sigma_X$ are constant parameters and $\xi$ is a standard Wiener process. The instantaneously stochastic returns on domestic and foreign assets are:

$$dR(t) = \eta dt + \sigma_R d\omega(t) \tag{A.2}$$

$$dR^*(t) = \eta dt + \sigma_{R^*} d\omega^*(t) \tag{A.3}$$

Here $R$ is the cumulative return on the domestic asset, $R^*$ the cumulative return on the foreign asset, and $\eta$ the expected return per unit time, assumed to be equal across countries and constant over time. The instantaneous correlation between the returns is denoted $\rho \equiv [E(d\omega d\omega^*)]/dt$. The correlations between the growth rate of nontradables consumption and the two asset returns are denoted $\rho_{RX} \equiv [E(d\omega d\xi)]/dt$ and $\rho_{R^*X} \equiv [E(d\omega^* d\xi)]/dt$.

The stochastic process for domestic financial wealth $W$ (measured in terms of traded goods) is

$$\begin{aligned} dW(t) = {} & [n(t)dR(t) + (1 - n(t))dR^*(t)]W(t) \\ & + p(t)S(t)dt - C(t)dt - p(t)X(t)dt \end{aligned} \tag{A.4}$$

where $C(t)$ is domestic consumption of tradables, $X(t)$ consumption of nontradables, $n(t)$ is the fraction of wealth invested at home and $p(t)$ is the relative price of nontradables.

The value function $V[W(t); X(t)]$ is defined as:

$$V[W(t); X(t)] \equiv \max_{C(t);n(t)} E_t \int_t^\infty e^{-\delta(s-t)} U[C(s); X(s)] ds \tag{A.5}$$

where $\delta$ is the rate of time preference. In equilibrium we have:

$$S(t) = X(t); \quad \frac{dS(t)}{S(t)} = \frac{dX(t)}{X(t)}$$

and the relative price of nontradables $p(t)$ is determined as the ratio of the marginal utilities $U_X/U_C$. Optimal consumption/portfolio choice $C(t)$ and $n(t)$ solves the Bellman equation:

$$\delta V[W(t);X(t)] = \max_{C(t);n(t)} U[C(t);X(t)] + (E_t dV[W(t);X(t)])/dt \quad \text{(A.6)}$$

subject to (A.1)-(A.4) and the appropriate boundary conditions.

Applying Ito's Lemma to the value function, we obtain

$$dV = V_X dX + V_W dW + \frac{1}{2}V_{XX} dX^2 + \frac{1}{2}V_{WW} dW^2 + V_{XW} dX dW. \quad \text{(A.7)}$$

Substituting $dX$ and $dW$ with their expressions (A.1) and (A.4), after taking the conditional expectation of (A.7) we can rewrite (A.6) as follows:

$$\begin{aligned}\delta V[W;X] = {} & \max_{C;n} U[C;X] + V_X X\mu + V_W(W\eta - C) + \frac{1}{2}V_{XX}X^2\sigma_X^2 \\ & + \frac{1}{2}V_{WW}W^2\left[n^2\sigma_R^2 + (1-n)^2\sigma_{R^*}^2 + 2n(1-n)\rho\sigma_R\sigma_{R^*}\right] \\ & + V_{WX}WX\sigma_X\left[n\rho_{RX}\sigma_R + (1-n)\rho_{R^*X}\sigma_{R^*}\right]\end{aligned}$$

Optimal portfolio shares are obtained by taking the first-order condition for a maximum with respect to $n$. Differentiating the previous expression yields:

$$\begin{aligned}0 = {} & V_{WW}W\left[n\sigma_R^2 - (1-n)\sigma_{R^*}^2 + (1-2n)\rho\sigma_R\sigma_{R^*}\right] \\ & + V_{WX}X\sigma_X\left[\rho_{RX}\sigma_R - \rho_{R^*X}\sigma_{R^*}\right] \quad \text{(A.8)}\end{aligned}$$

from which we derive the expressions for $n$ and bias in the main text.

The model can be generalized in several respects. Consider first the case in which expected returns are not constant over time, but rather exhibit mean reversion around a constant steady-state mean. Specifically, replace (A.2) and (A.3) with

$$dR(t) = \eta(t)\,dt + \sigma_R d\omega(t) \quad \text{(A.9)}$$
$$dR^*(t) = \eta^*(t)\,dt + \sigma_{R^*} d\omega^*(t) \quad \text{(A.10)}$$

where

$$d\eta(t) = -k(\eta(t) - \mu)\,dt + \phi\, d\omega(t)$$
$$d\eta^*(t) = -k(\eta^*(t) - \mu)\,dt + \phi^* d\omega^*(t)$$

are two Ornstein-Uhlenbeck processes, with $\mu$, $\phi$ and $\phi^*$ constant positive parameters. Intuitively, a shock $d\omega$ pushes upward the current return and the future expected return on domestic assets. If the expected return $\eta$ is currently above its long-run average $\mu$, it is expected to decline; if $\eta < \mu$ instead, $\eta$ is expected to raise. In the long run, the distribution for the expected return $\eta$ is stationary, and $\eta$ is normally distributed with asymptotic mean $\mu$ and asymptotic variance $\phi^2/2k$. Similar considerations hold for $\mu^*$.

Under these new assumptions, the value function includes now $\eta$ and $\eta^*$ as arguments, and the modified Bellman equation is now:

$$\begin{aligned}
\delta V[W; X; \eta; \eta^*] = \max_{C;n} & \; U[C; X] + V_X X \mu + V_W [W [n\eta + (1-n)\eta^*] - C] \\
& - V_\eta k(\eta - \mu) - V_{\eta^*} k(\eta^* - \mu) + \frac{1}{2} V_{XX} X^2 \sigma_X^2 \\
& + \frac{1}{2} V_{WW} W^2 \left[ n^2 \sigma_R^2 + (1-n)^2 \sigma_{R^*}^2 + 2n(1-n)\rho\sigma_R\sigma_{R^*} \right] \\
& + \frac{1}{2} V_{\eta\eta}\phi^2 + \frac{1}{2} V_{\eta^*\eta^*}(\phi^*)^2 + V_{WX} W X \sigma_X [n\rho_{RX}\sigma_R + (1-n)\rho_{R^*X}\sigma_{R^*}] \\
& + V_{W\eta} W \phi [n\sigma_R + (1-n)\rho\sigma_{R^*}] + V_{W\eta^*} W \phi^* [n\rho\sigma_R + (1-n)\sigma_{R^*}] \\
& + V_{X\eta} X \sigma_X \phi \rho_{RX} + V_{X\eta^*} X \sigma_X \phi^* \rho_{R^*X} + V_{\eta\eta^*}\phi\phi^*\rho
\end{aligned}$$

The first order condition with respect to $n$ now yields

$$\begin{aligned}
0 = & \; V_W(\eta - \eta^*) + V_{WW} W \left[ n\sigma_R^2 - (1-n)\sigma_{R^*}^2 + (1-2n)\rho\sigma_R\sigma_{R^*} \right] \\
& + V_{WX} X \sigma_X [\rho_{RX}\sigma_R - \rho_{R^*X}\sigma_{R^*}] + V_{W\eta}\phi[\sigma_R - \rho\sigma_{R^*}] + V_{W\eta^*}\phi^*[\rho\sigma_R - \sigma_{R^*}]
\end{aligned}$$

so that the portfolio share invested in domestic assets is equal to

$$\begin{aligned}
n = & \; \frac{\sigma_{R^*}^2 - \rho\sigma_R\sigma_{R^*}}{\sigma_{R^*}^2 + \sigma_R^2 - 2\rho\sigma_R\sigma_{R^*}} - \frac{V_W}{V_{WW}W}\frac{\eta - \eta^*}{\sigma_{R^*}^2 + \sigma_R^2 - 2\rho\sigma_R\sigma_{R^*}} \\
& - \frac{V_{W\eta}}{V_{WW}W}\frac{\phi[\sigma_R - \rho\sigma_{R^*}]}{\sigma_{R^*}^2 + \sigma_R^2 - 2\rho\sigma_R\sigma_{R^*}} - \frac{V_{W\eta^*}}{V_{WW}W}\frac{\phi^*[\rho\sigma_R - \sigma_{R^*}]}{\sigma_{R^*}^2 + \sigma_R^2 - 2\rho\sigma_R\sigma_{R^*}} + \text{bias},
\end{aligned}$$

where bias is the same formula given in the main text.

Interpreting the formula above, the share of wealth invested in domestic assets is equal to the minimum-variance portfolio share (first term), corrected to account for discrepancies in expected returns (second term), to hedge against uncertainty in the expected returns themselves (third and fourth term), and to hedge against nontradables uncertainty (the bias term). Only the last term is associated with hedging against nontradables. The first four terms are present even in the absence of nontradables. The key point is that a higher expected return of one country's assets leads investors of all countries to shift their portfolio toward that country's assets, and therefore leads neither to a systematic difference in portfolio shares across investors in different countries, nor to a systematic tendency to skew investors' portfolios toward the assets issued in their countries of residence. Such systematic tendency is still captured by the term bias, which measures the impact of country-specific variables on wealth allocation, although theoretically the presence of nontradables may also have a small secondary impact on the composition of investors' portfolios through their effects on the preference terms $V_{W\eta}/V_{WW}W$ and $V_{W\eta^*}/V_{WW}W$.

Similar considerations apply in the case of stochastic volatility. Assuming for simplicity of exposition that only domestic asset returns are heteroskedastic, the processes for asset returns are now

$$dR(t) = \eta dt + \sigma_R(t)\, d\omega(t) \quad \text{(A.11)}$$
$$dR^*(t) = \eta dt + \sigma_{R^*} d\omega^*(t) \quad \text{(A.12)}$$

where the law of motion for $\sigma_R(t)$ is given by:

$$d\sigma_R(t) = -k[\sigma_R(t) - \bar{\sigma}_R]dt + \phi\sqrt{\sigma_R(t)}dz(t);$$

In other words, $\sigma_R$ follows an autoregressive process similar to the one considered above for expected returns: the conditional volatility of domestic returns is subject to shocks $dz$ and is time-varying around a steady-state average $\bar{\sigma}_R$. Deviations from the steady-state average decay at the rate $k$. Assuming that at some initial time $t = 0$ it is $\sigma_R(0) > 0$, the square root term in the previous equation guarantees that $\sigma_R$ is always positive at any time $t$. We denote the correlations involving $\sigma_R$ as $\rho_{\sigma R} \equiv [E(d\omega dz)]/dt$, $\rho_{\sigma R^*} \equiv [E(d\omega^* dz)]/dt$ and $\rho_{\sigma X} \equiv [E(dz d\xi)]/dt$.

The modified Bellman equation is now:

$$\begin{aligned} \delta V[W;X;\sigma_R] &= \max_{C;n} U[C;X] + V_X X\mu + V_W[W\eta - C] \\ &\quad - V_{\sigma_R} k(\sigma_R - \bar{\sigma}_R) + \frac{1}{2}V_{XX}X^2\sigma_X^2 \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}V_{WW}W^2\Big[n^2\sigma_R^2 + (1-n)^2\sigma_{R^*}^2 + 2n(1-n)\rho\sigma_R\sigma_{R^*}\Big]\\
&+\frac{1}{2}V_{\sigma_R\sigma_R}\phi^2\sigma_R + V_{WX}WX\sigma_X\,[n\rho_{RX}\sigma_R + (1-n)\rho_{R^*X}\sigma_{R^*}]\\
&+V_{W\sigma_R}W\phi\sqrt{\sigma_R}\,[n\sigma_R\rho_{\sigma_R} + (1-n)\sigma_{R^*}\rho_{\sigma_{R^*}}]\\
&+V_{X\sigma_R}X\sigma_X\rho_{\sigma_X}\phi\sqrt{\sigma_R}
\end{aligned}
$$

and taking the first-order condition, the portfolio share invested in domestic assets is equal to

$$
n = \frac{\sigma_{R^*}^2 - \rho\sigma_R\sigma_{R^*}}{\sigma_{R^*}^2 + \sigma_R^2 - 2\rho\sigma_R\sigma_{R^*}} - \frac{V_{W\sigma_R}}{V_{WW}W}\frac{\phi\sqrt{\sigma_R}\,[\sigma_R\rho_{\sigma_R} - \sigma_{R^*}\rho_{\sigma_{R^*}}]}{\sigma_{R^*}^2 + \sigma_R^2 - 2\rho\sigma_R\sigma_{R^*}} + \text{bias}.
$$

As in the case with time-varying asset returns, we now have an additional hedging component related to stochastic volatility. The first two terms are present even in the absence of nontradables. All investors, no matter where they are located, prefer to invest in assets whose return rises when volatility increases. The term bias, on the other hand, captures systematic discrepancies in national portfolios as the result of optimal hedging behavior. We do want to be somewhat cautionary though by again pointing out that nontradables may have a small secondary impact on portfolios through their effect on the preference term $V_{W\sigma_R}/V_{WW}W$.

# 2 The sign of $\tau'$ without balanced growth

In the context of a deterministic framework we want to show that the sign of $\tau'$ is opposite to that of the growth rate of the fraction spent on nontradables.

Without uncertainty, the differential equation for $\tau$, described in Appendix A, becomes:

$$
\left(1 - \frac{\tau'^{\ddagger}}{\tau}\right)(\tau - \eta + \mu) = \frac{1+m}{1+\gamma\epsilon m}\epsilon(\delta + \gamma\mu - \eta) \qquad \text{(B.1)}
$$

It follows that

$$
\begin{aligned}
\frac{dX}{Xdt} - \frac{dC}{Cdt} &= \mu - \frac{d\tau}{\tau dt} - \frac{dW}{Wdt} = \mu - \frac{\tau'^{\ddagger}}{\tau}(\mu - \eta + \tau) - \eta + \tau =\\
&= \left(1 - \frac{\tau'^{\ddagger}}{\tau}\right)(\mu - \eta + \tau) = \frac{1+m}{1+\gamma\epsilon m}\epsilon(\delta + \gamma\mu - \eta) \qquad \text{(B.2)}
\end{aligned}
$$

Therefore, $dC/C > dX/X$ if and only if $\eta > \delta + \gamma\mu$.

Assume that the latter condition holds. Since $m = (1 - \alpha)\alpha^{-1}(C/X)^{1-1/\epsilon} = C/pX$, it follows that the fraction spent on nontradables $1/(1 + m)$ rises (falls) over time if $\epsilon < 1$ ($\epsilon > 1$). Moreover, since

$$\frac{dC}{Cdt} = \mu - \frac{1 + m}{1 + \gamma\epsilon m}\epsilon(\delta + \gamma\mu - \eta); \tag{B.3}$$

it follows that $\partial(dC/Cdt)/\partial m > 0$ when the two consumption goods are complementary in preferences ($\epsilon\gamma < 1$). Therefore,

$$\frac{\partial(dC/Cdt)}{\partial(C/X)} = \frac{\partial(dC/Cdt)}{\partial m}\frac{\partial m}{\partial(C/X)} =$$

$$= -\left(\frac{1-\alpha}{\alpha}\right)\frac{(1-\gamma\epsilon)(\delta+\gamma\mu-\eta)(\epsilon-1)}{(1+\gamma\epsilon m)^2}\left(\frac{C}{X}\right)^{-\frac{1}{\epsilon}} \tag{B.4}$$

is negative (positive) when $\epsilon < 1$ ($\epsilon > 1$).

It is now easy to check that the sign of $\tau'$ is the opposite of that of the growth rate of the fraction spent on nontradables. We show this for $\epsilon < 1$; so that the growth rate of tradables consumption is a negative function of the ratio $C/X$. Consider the impact of an unanticipated permanent increase in $X$ occurring at time 0. Conjecture first that $\tau' = 0$. In this case, tradables consumption at time 0 does not react to the change in $X$. However, the fall in the $C/X$ ratio instantaneously translates into a higher growth rate of tradables consumption.

Although the growth rate of tradables consumption decelerates in the future, the level of tradables consumption is always higher than before the shock to $X$. In fact, suppose that at some future time $\tilde{t}$ the after-shock path of $C$ crosses the before-shock path. Necessarily, at this future date $C/X$ is lower than it would have been before the shock. But this implies that at $\tilde{t}$ the growth rate of tradables consumption must be higher than the growth rate before the shock, so that the two consumption paths can never intersect. As a result, tradables consumption is always higher after the shock to $X$. Since financial wealth is not affected by the shock, the budget constraint is violated.

Suppose now that $\tau' > 0$. At time 0, tradables consumption jumps upward in response to the rise in $X$. The logic is the same as above: if there existed a time $\tilde{t}$ at which the old and new path of C met, at that time the consumption ratio $C/X$ would be lower than before the shock, and $dC/C$ would be paradoxically higher. As C is permanently above the old path, once again the budget constraint is violated. Only the case $\tau' < 0$ does not entail logical inconsistencies or violations of the transversality condition. One can use similar arguments for $\epsilon > 1$ and for $\eta < \delta + \gamma\mu$, to show that the sign of $\tau'$ is always the opposite of that of the growth rate of the fraction spent on nontradables.

# 3 Small levels of risk

We want to show that under balanced growth, a marginal increase in the standard deviation of X from zero has a negligible effect on $\tau$ and $\tau'$ as they depend on the variance of X rather than the standard deviation.

**Proposition.** Let $\mu = \gamma^{-1}(\eta - \delta)$ and $\delta - (1-\gamma)\mu > 0$. Holding $\rho$, $\rho_{RX}$, $\rho_{R^*X}$, $\sigma_X/\sigma_R$, and $\sigma_X/\sigma_{R^*}$ constant, $\partial\tau/\partial\sigma_X^2$ and $\partial\tau'/\partial\sigma_X^2$, both evaluated at $\sigma_X^2 = 0$, are finite functions of X and W.

**Proof.** Recalling the definition of $\Sigma$ and $\sigma$ in Appendix A, define $\hat{\Sigma} \equiv \Sigma/\sigma_X^2$, $\hat{\sigma} \equiv \sigma/\sigma_X^2$; and hold $\hat{\Sigma}$ and $\hat{\sigma}$ constant with respect to changes in $\sigma_X$. After substitution of n from equation (A.8) back into the Bellman equation, the latter becomes:

$$\delta V = U(C;X) + V_W W\eta - V_W C + V_X X\mu + A(W;X;\sigma_X^2)\sigma_X^2$$

where

$$A(W;X;\sigma_X^2) \equiv \frac{1}{2}\frac{V_{WW}W^2}{\iota'\hat{\Sigma}^{-1}\iota} + \frac{V_{WX}WX\hat{\sigma}'\hat{\Sigma}^{-1}\iota}{\iota'\hat{\Sigma}^{-1}\iota} + \frac{1}{2}V_{XX}X^2 - \frac{1}{2}\frac{V_{WX}^2X^2}{V_{WW}}\left[\hat{\sigma}'\hat{\Sigma}^{-1}\hat{\sigma} - \frac{\left(\hat{\sigma}'\hat{\Sigma}^{-1}\iota\right)^2}{\iota'\hat{\Sigma}^{-1}\iota}\right].$$

Now take the right hand side differential of the Bellman equation with respect to $\sigma_X^2$, evaluated at $\sigma_X^2 = 0$. Using (i) the envelope condition $U_C = V_W$, (ii) the homogeneous function property $V_W W + V_X X =$

$(1-\gamma)V$, (iii) the balanced growth condition $\mu = \frac{1}{\gamma}(\eta - \delta)$; and (iv) the fact that without uncertainty $\tau = \frac{1}{\gamma}\delta + (1 - \frac{1}{\gamma})\eta = \eta - \mu$, we get:

$$V_{\sigma_X^2} = a_0 A(W; X)$$

where $a_0 = (\delta - (1-\gamma)\mu)^{-1}$ and $A(W; X)$ is shorthand for $A(W; X; 0)$. Differentiating the equation above with respect to $W$, and using the envelope condition $V_W = U_C$, we get:

$$aA_W = V_{W\sigma_X^2} = U_{CC}\frac{\partial \tau}{\partial \sigma_X^2}W$$

which implies

$$\frac{\partial \tau}{\partial \sigma_X^2} = \frac{aA_W}{U_{CC}W}.$$

Similarly, differentiating with respect to both $X$ and $W$ yields

$$aA_{WX} = V_{WX\sigma_X^2} = \frac{\partial}{\partial \sigma_X^2}\frac{\partial U_C}{\partial X} =$$

$$\frac{\partial}{\partial \sigma_X^2}(U_{CC}\tau' + U_{CX}) = U_{CC}\frac{\partial \tau'}{\partial \sigma_X^2} + U_{CCX}W\frac{\partial \tau}{\partial \sigma_X^2}$$

which implies

$$\frac{\partial \tau'}{\partial \sigma_X^2} = \frac{aA_{WX}}{U_{CC}} - \frac{aA_W U_{CCX}}{U_{CC}^2}$$

This proves that both $\partial \tau / \partial \sigma_X^2$ and $\partial \tau' / \partial \sigma_X^2$, evaluated at $\sigma_X^2 = 0$, are finite functions of $X$ and $W$.